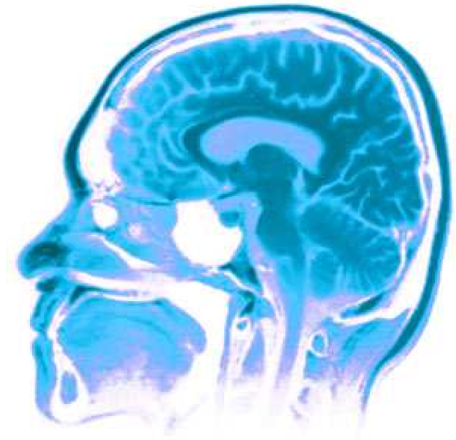
CPSC540



Bayesian learning

Nando de Freitas
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University of British Columbia

Outline of the lecture

This lecture introduces **Bayes rule** and Bayesian learning for linear models.

The goal is for you to:

- ❑ Learn how Bayes rule is derived.
- ❑ Learn to apply Bayes rule to simple examples.
- ❑ Learn how to apply Bayesian learning to linear models.
- ❑ Learn the mechanics of conjugate analysis.

Problem 1: Diagnoses

- The doctor has bad news and good news.

- The bad news is that you tested positive for a serious disease, and that **the test is 99% accurate** (i.e., the probability of testing positive given that you have the disease is 0.99, as is the probability of testing negative given that you don't have the disease).

- The good news is that this is a rare disease, striking only 1 in 10,000 people.

- What are the chances that you actually have the disease?

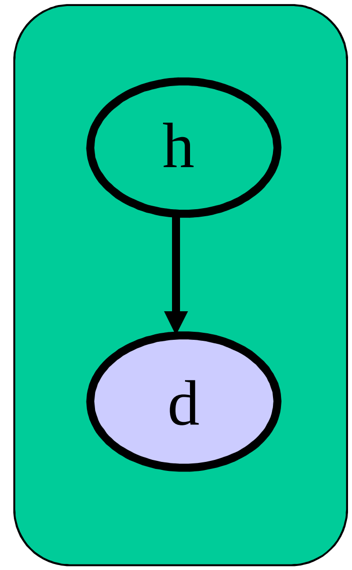
Bayes rule

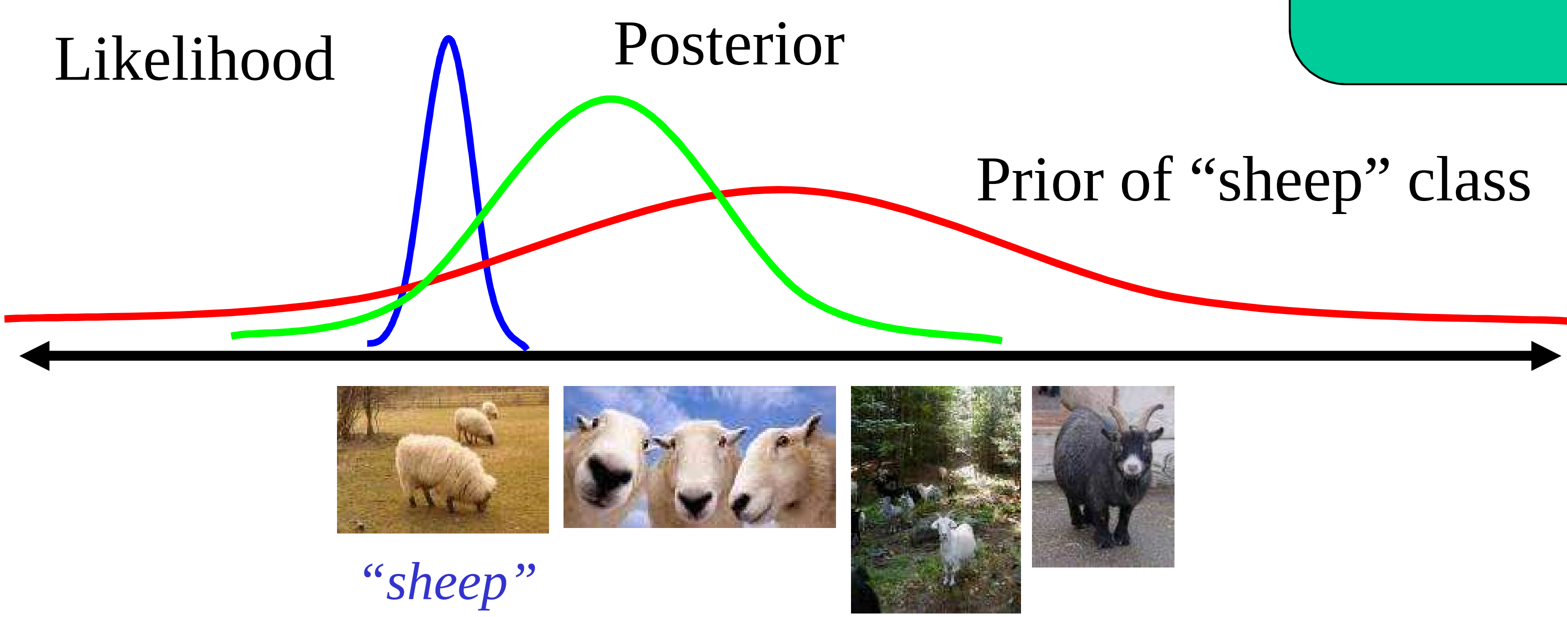
Bayes rule enables us to reverse probabilities:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Learning and Bayesian inference

$$p(h \mid d) = \frac{p(d \mid h)\, p(h)}{\sum_{h' \in H} p(d \mid h')\, p(h')}$$





Problem 1: Diagnoses

The test is 99% accurate: $P(T=1|D=1)$ *= 0.99 and* $P(T=0|D=0)$ *= 0.99*
Where T *denotes test and* D *denotes disease.*

The disease affects 1 in 10000: $P(D=1)$ *= 0.0001*

$$P(D=1|T=1) = \frac{P(T=1|D=1)P(D=1)}{P(T=1|D=0)P(D=0)+P(T=1|D=1)P(D=1)}$$

$$=$$

Speech recognition

$$P(\text{words} \mid \text{sound}) \;\; \alpha \;\; P(\text{sound} \mid \text{words})\, P(\text{words})$$

Final beliefs

Likelihood of data

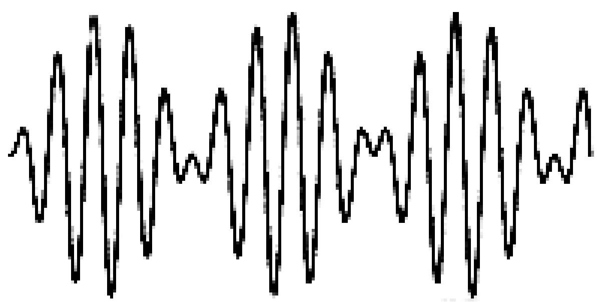
eg mixture of Gaussians

Prior language model

eg unigrams

Hidden Markov Model (HMM)

“Recognize speech”



“Wreck a nice beach”

Bayesian learning for model parameters

Step 1: *Given* n *data,* $D = x_{1:n} = \{x_1, x_2, \ldots, x_n\}$*, write down the expression for the* ***likelihood****:*

$$p(D \mid \theta)$$

Step 2: *Specify a* ***prior****:* $p(\theta)$

Step 3: *Compute the* ***posterior****:*

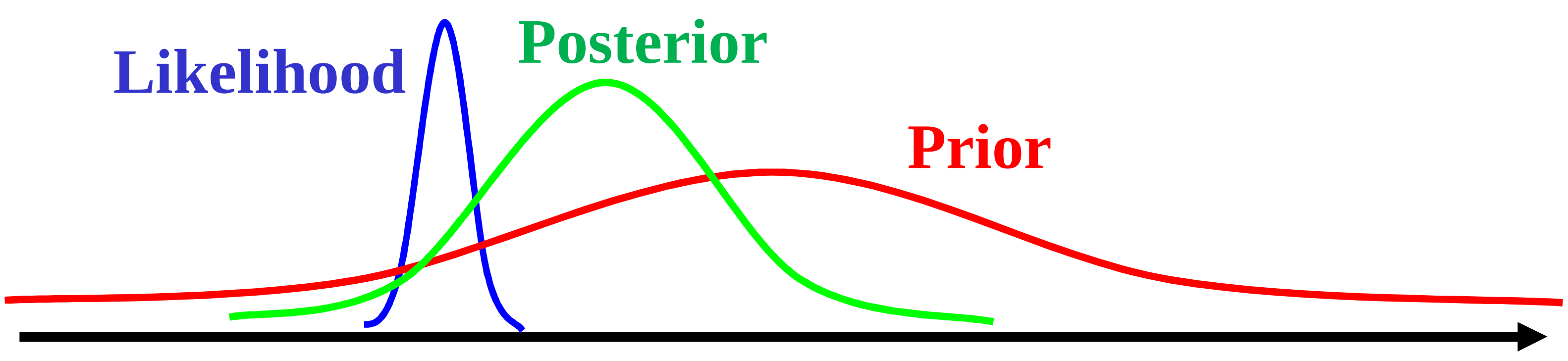
$$p(\theta \mid D) = \frac{p(D \mid \theta)\, p(\theta)}{p(D)}$$

Bayesian linear regression

The likelihood is a Gaussian, $\mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\theta}, \sigma^2\mathbf{I}_n)$. The conjugate prior is also a Gaussian, which we will denote by $p(\boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{\theta}|\boldsymbol{\theta}_0, \mathbf{V}_0)$.

Using Bayes rule for Gaussians, the posterior is given by

$$
\begin{aligned}
p(\boldsymbol{\theta}|\mathbf{X}, \mathbf{y}, \sigma^2) &\propto \mathcal{N}(\boldsymbol{\theta}|\boldsymbol{\theta}_0, \mathbf{V}_0)\mathcal{N}(\mathbf{y}|\mathbf{X}\boldsymbol{\theta}, \sigma^2\mathbf{I}_n) = \mathcal{N}(\boldsymbol{\theta}|\boldsymbol{\theta}_n, \mathbf{V}_n) \\
\boldsymbol{\theta}_n &= \mathbf{V}_n\mathbf{V}_0^{-1}\boldsymbol{\theta}_0 + \frac{1}{\sigma^2}\mathbf{V}_n\mathbf{X}^T\mathbf{y} \\
\mathbf{V}_n^{-1} &= \mathbf{V}_0^{-1} + \frac{1}{\sigma^2}\mathbf{X}^T\mathbf{X}
\end{aligned}
$$



Bayesian linear regression

Assume σ^2 is known.

$$P(\theta \mid X, Y, \sigma^2) \propto P(Y \mid X, \theta, \sigma^2)\, P(\theta)$$

$$\propto e^{-\frac{1}{2}(Y-X\theta)^T(\sigma^2 I)^{-1}(Y-X\theta)}\; e^{-\frac{1}{2}(\theta-\theta_0)^T V_0^{-1}(\theta-\theta_0)}$$

$$= e^{-\frac{1}{2}\left\{ Y^T(\sigma^2 I)^{-1}Y - 2Y^T(\sigma^2 I)^{-1}X\theta + \theta^T X^T(\sigma^2 I)^{-1}X\theta + \theta^T V_0^{-1}\theta + \theta_0^T V_0^{-1}\theta_0 - 2\theta_0^T V_0^{-1}\theta \right\}}$$

$$= e^{-\frac{1}{2}\left\{ \text{const} + \theta^T \underbrace{\left(X^T(\sigma^2 I)^{-1}X + V_0^{-1}\right)}_{\text{Call this } V_n^{-1}}\theta - 2\left(Y^T(\sigma^2 I)^{-1}X + \theta_0^T V_0^{-1}\right)\theta \right\}}$$

$$= e^{-\frac{1}{2}\left\{ \text{const} + \theta^T V_n^{-1}\theta - 2\left(\frac{Y^T X}{\sigma^2} + \theta_0^T V_0^{-1}\right)\theta \right\}}$$

$$= e^{-\frac{1}{2}\left\{ \text{const} + \theta^T V_n^{-1}\theta - 2\theta_n^T V_n^{-1}\theta + 2\theta_n^T V_n^{-1}\theta - 2\left(\frac{Y^T X}{\sigma^2} + \theta_0^T V_0^{-1}\right)\theta \right\}}$$

$$= e^{-\frac{1}{2}\left\{ \text{const}_2 + (\theta-\theta_n)^T V_n^{-1}(\theta-\theta_n) + 2\left[\theta_n^T V_n^{-1} - \frac{Y^T X}{\sigma^2} - \theta_0^T V_0^{-1}\right]\theta \right\}}$$

Bayesian linear regression

$$\theta_n^T V_n^{-1} - \frac{y^T X}{\sigma^2} - \theta_0^T V_0^{-1} = 0 \quad \text{when} \quad \theta_n = V_n \left[V_0^{-1}\theta_0 + \frac{X^T y}{\sigma^2} \right]$$

and when this happens, we have:

$$P(\theta \mid X, y, \sigma^2) \propto e^{-\frac{1}{2}(\theta - \theta_n) V_n^{-1} (\theta - \theta_n)}$$

By the definition of a multivariate Gaussian, we have:

$$\int e^{-\frac{1}{2}(\theta-\theta_n)^T V_n^{-1}(\theta-\theta_n)} \, d\theta = |2\pi V_n|^{1/2}$$

$$\therefore \quad P(\theta \mid X, y, \sigma^2) = |2\pi V_n|^{-1/2} e^{-\frac{1}{2}(\theta-\theta_n)^T V_n^{-1}(\theta-\theta_n)}$$

$\blacksquare$

Bayesian linear regression

Consider the special case where $\boldsymbol{\theta}_0 = \mathbf{0}$ and $\mathbf{V}_0 = \tau_0^2 \mathbf{I}_d$, which is a spherical Gaussian prior. Then the posterior mean reduces to

$$\begin{aligned}
\boldsymbol{\theta}_n &= \frac{1}{\sigma^2}\mathbf{V}_N \mathbf{X}^T \mathbf{y} = \frac{1}{\sigma^2}\left(\frac{1}{\tau_0^2}\mathbf{I}_d + \frac{1}{\sigma^2}\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y} \\
&= \left(\lambda \mathbf{I}_d + \mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}
\end{aligned}$$

where we have defined $\lambda := \frac{\sigma^2}{\tau_0^2}$. We have therefore recovered **ridge regression** again!

Bayesian versus ML plugin prediction

Posterior mean: $\theta_n = \left(\lambda \mathbf{I}_d + \mathbf{X}^T \mathbf{X}\right)^{-1} \mathbf{X}^T \mathbf{y}$

Posterior variance: $V_n = \sigma^2 \left(\lambda \mathbf{I}_d + \mathbf{X}^T \mathbf{X}\right)^{-1}$

To predict, Bayesians marginalize over the posterior. Let x_* be a new input. The prediction, given the training data $D=(X, y)$, is:

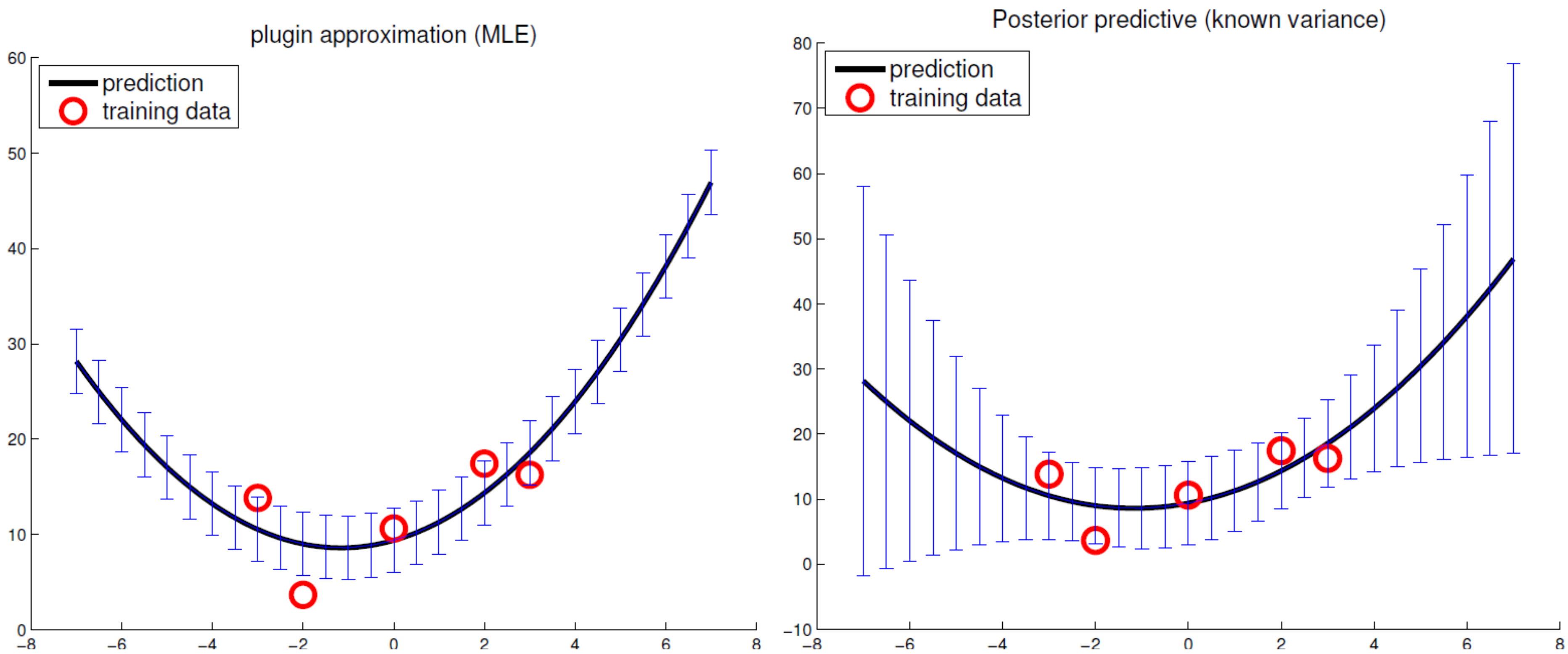
$$P(y \mid x_*, D, \sigma^2) = \int \mathcal{N}(y \mid x_*^T \theta, \sigma^2)\, \mathcal{N}(\theta \mid \theta_n, V_n)\, d\theta$$

$$= \mathcal{N}(y \mid x_*^T \theta_n, \sigma^2 + x_*^T V_n x_*)$$

On the other hand, the ML plugin predictor is:

$$P(y \mid x_*, D, \sigma^2) = \mathcal{N}(y \mid x_*^T \theta_{ML}, \sigma^2)$$

Bayesian versus ML plug-in prediction



Next lecture

In the next lecture, we extend Bayesian learning to nonlinear problems via a technique known as Gaussian processes.