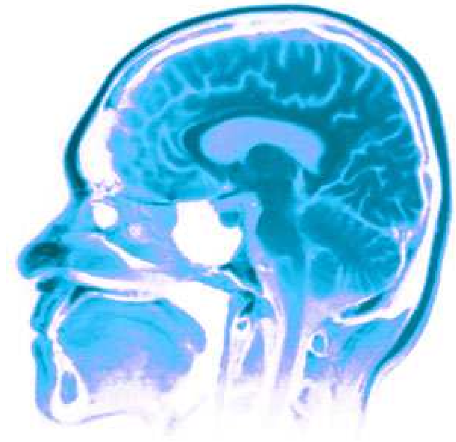
CPSC540



Neural Networks

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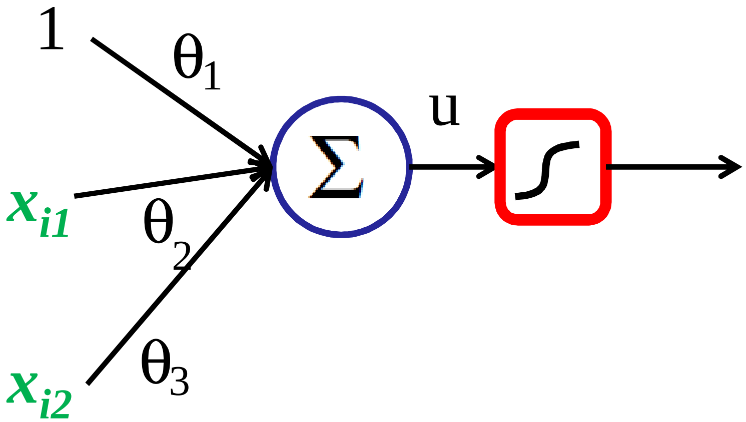
University of British Columbia

Outline of the lecture

This lecture introduces you to the fascinating subject of classification and regression with artificial neural networks. In particular, it

- ❑ Introduces multi-layer perceptrons (MLPs)
- ❑ Teaches you how to combine probability with neural networks so that the nets can be applied to regression, binary classification and multivariate classification.
- ❑ Describes the relation between energy functions (cost/loss functions) and probabilistic models.

MLP – 1 neuron

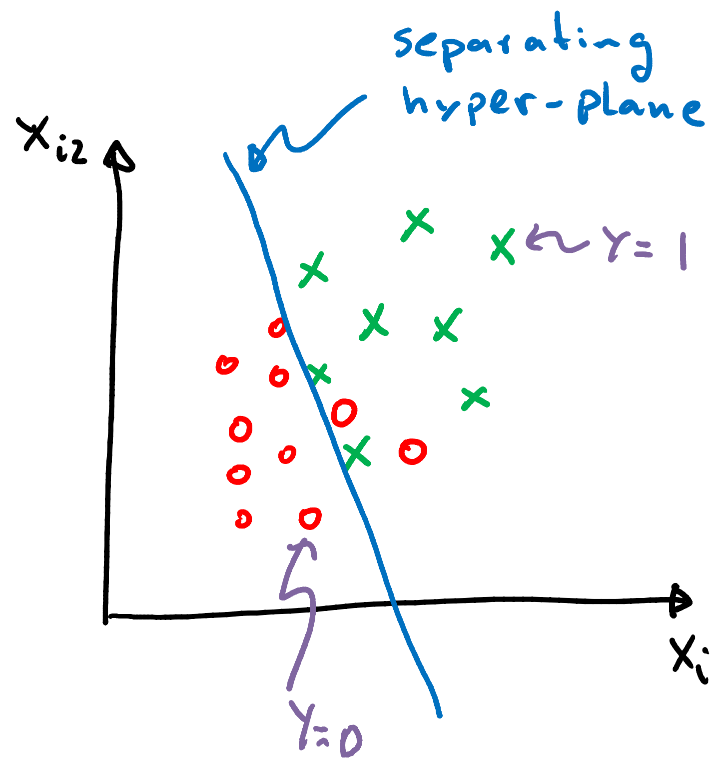


$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

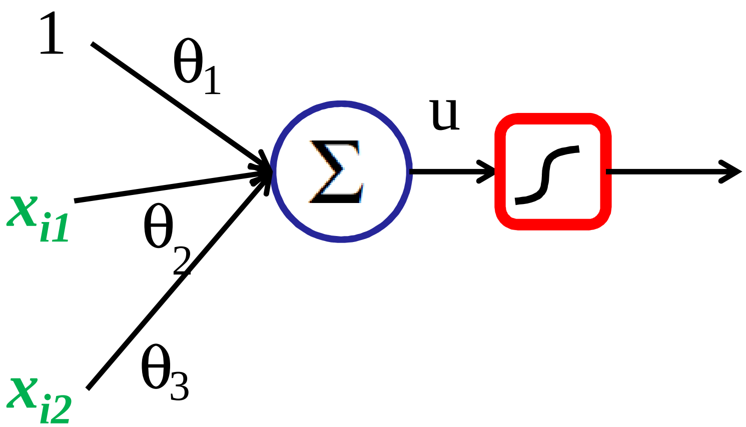
We are given the data $\{x_i, y_i\}_{i=1}^{n}$

eg.

	x_{i1}	x_{i2}	y_i
$i=1$	0.2	6	0
$i=2$	0.3	22	1
$i=3$	0.6	-0.6	1
$i=4$	-0.4	58	0
⋮			



MLP – 1 neuron



$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

$$0 < \hat{y}_i < 1$$

$$\boxed{u = \theta_1 + \theta_2 x_{i1} + \theta_3 x_{i2}}$$

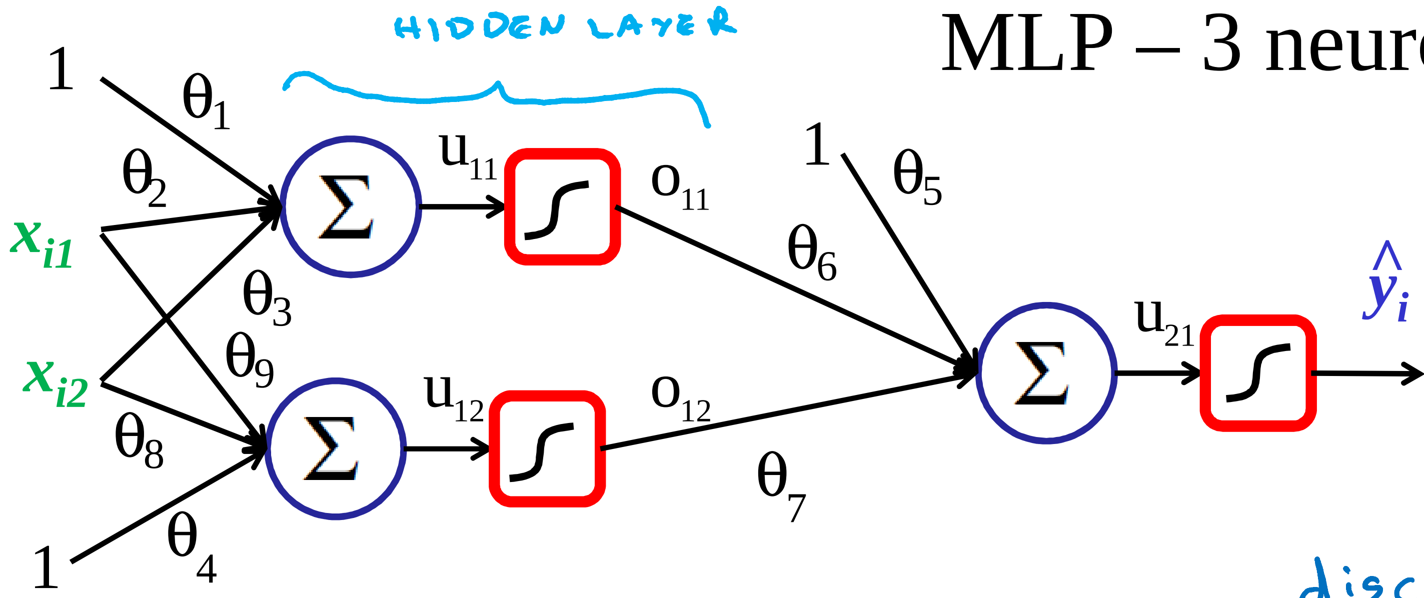
$$\hat{y}_i = \frac{1}{1+e^{-\boxed{u}}} = \frac{1}{1+e^{-\theta_1-\theta_2 x_{i1} - \theta_3 x_{i2}}} = P(y_i = 1 \mid \underline{x}_i, \underline{\theta})$$

$$P(y_i \mid \underline{x}_i, \underline{\theta}) = \hat{y}_i^{\,y_i} (1-\hat{y}_i)^{1-y_i} = \begin{cases} \hat{y}_i & \text{when } y_i = 1 \\ 1-\hat{y}_i & \text{otherwise} \end{cases}$$

For n independent observations (Bernoulli)

$$P(\underline{y} \mid \underline{\underline{x}}, \underline{\theta}) = \prod_{i=1}^{n} P(y_i \mid \underline{x}_i, \underline{\theta})$$

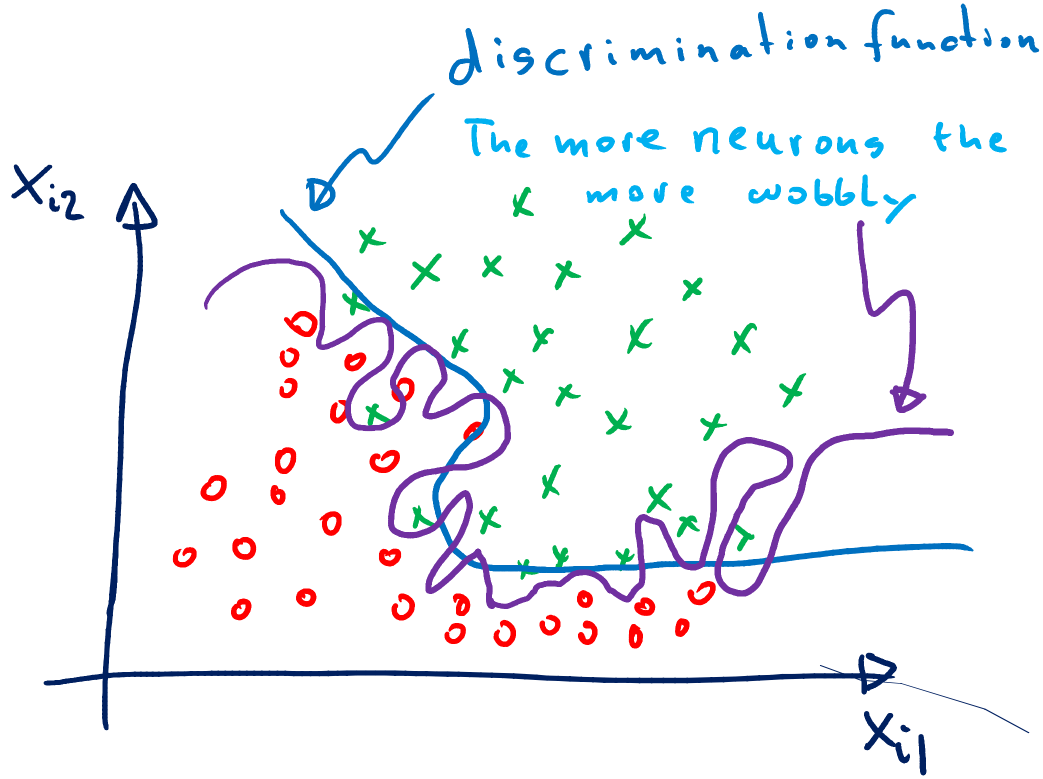
MLP – 3 neurons, 2 layers



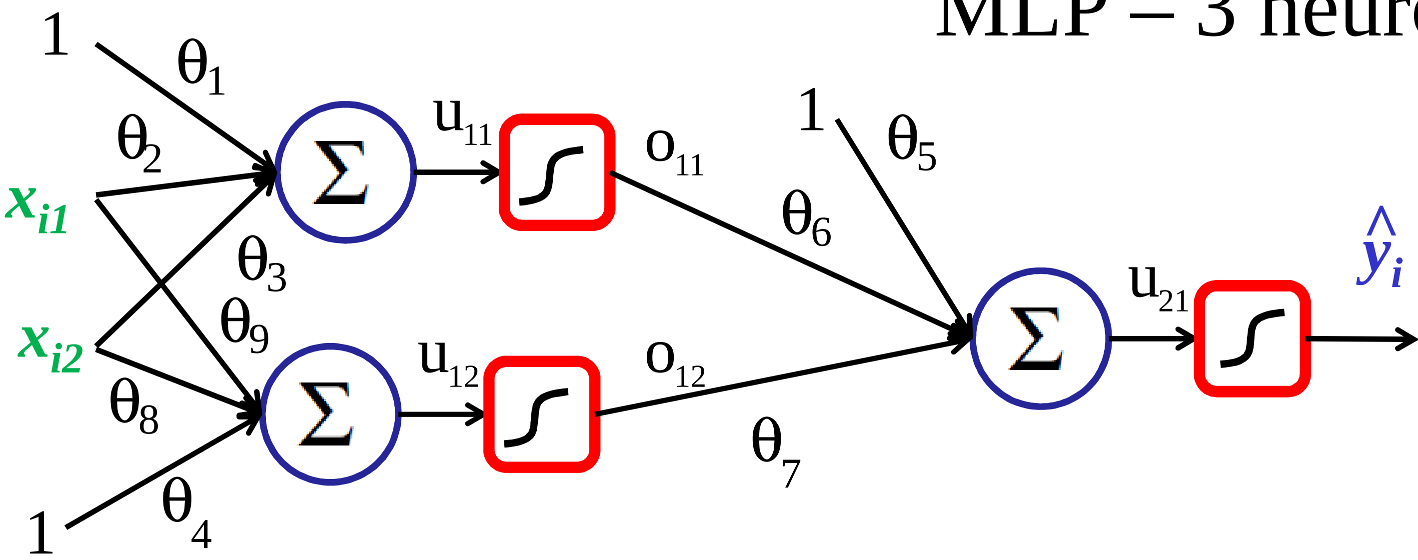
$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

Data:

	x_{i1}	x_{i2}	y_i
i=1	6	8	1
i=2	0.2	-5	0
	-100	3.1	1
	6	9	0
	5	8	0



MLP – 3 neurons, 2 layers



$$\hat{y}_i = P(y_i = 1 \mid x_i, \theta)$$

$$u_{11} = \theta_1 + \theta_2 x_{i1} + \theta_3 x_{i2}$$

$$o_{11} = 1/_{1+e^{-u_{11}}}$$

$$\hat{y}_i = \frac{1}{1+e^{-u_{21}}}$$

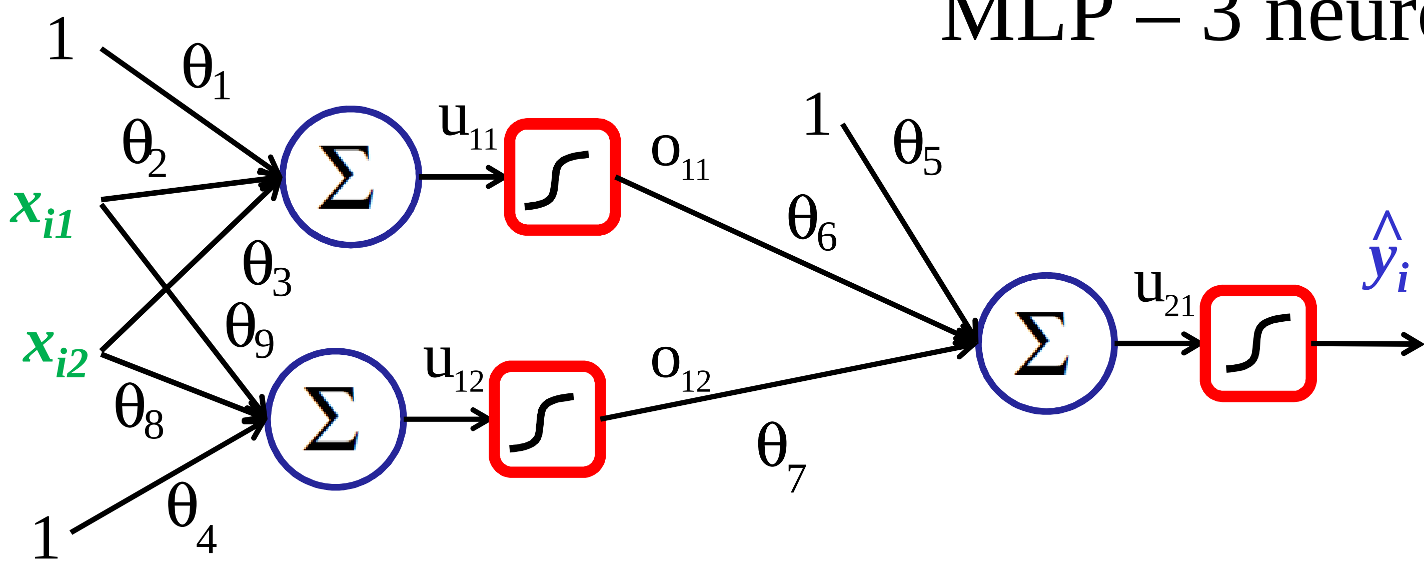
$$u_{12} = \theta_4 + \theta_9 x_{i1} + \theta_8 x_{i2}$$

$$o_{12} = 1/_{1+e^{-u_{12}}}$$

$$u_{21} = \theta_5 + \theta_6 o_{11} + \theta_7 o_{12}$$

$$P(y_i \mid \underline{x}_i, \underline{\theta}) = \hat{y}_i^{\,y_i} (1 - \hat{y}_i)^{1-y_i}$$

MLP – 3 neurons, 2 layers



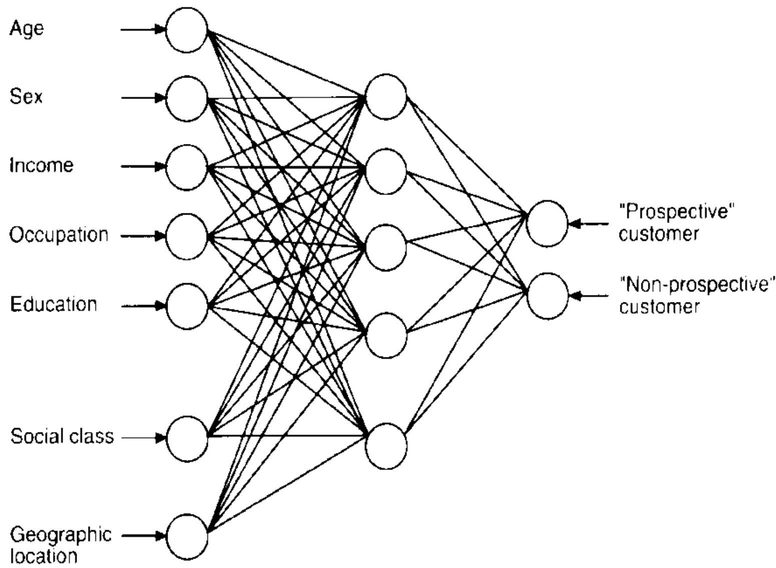
For n independent observations.

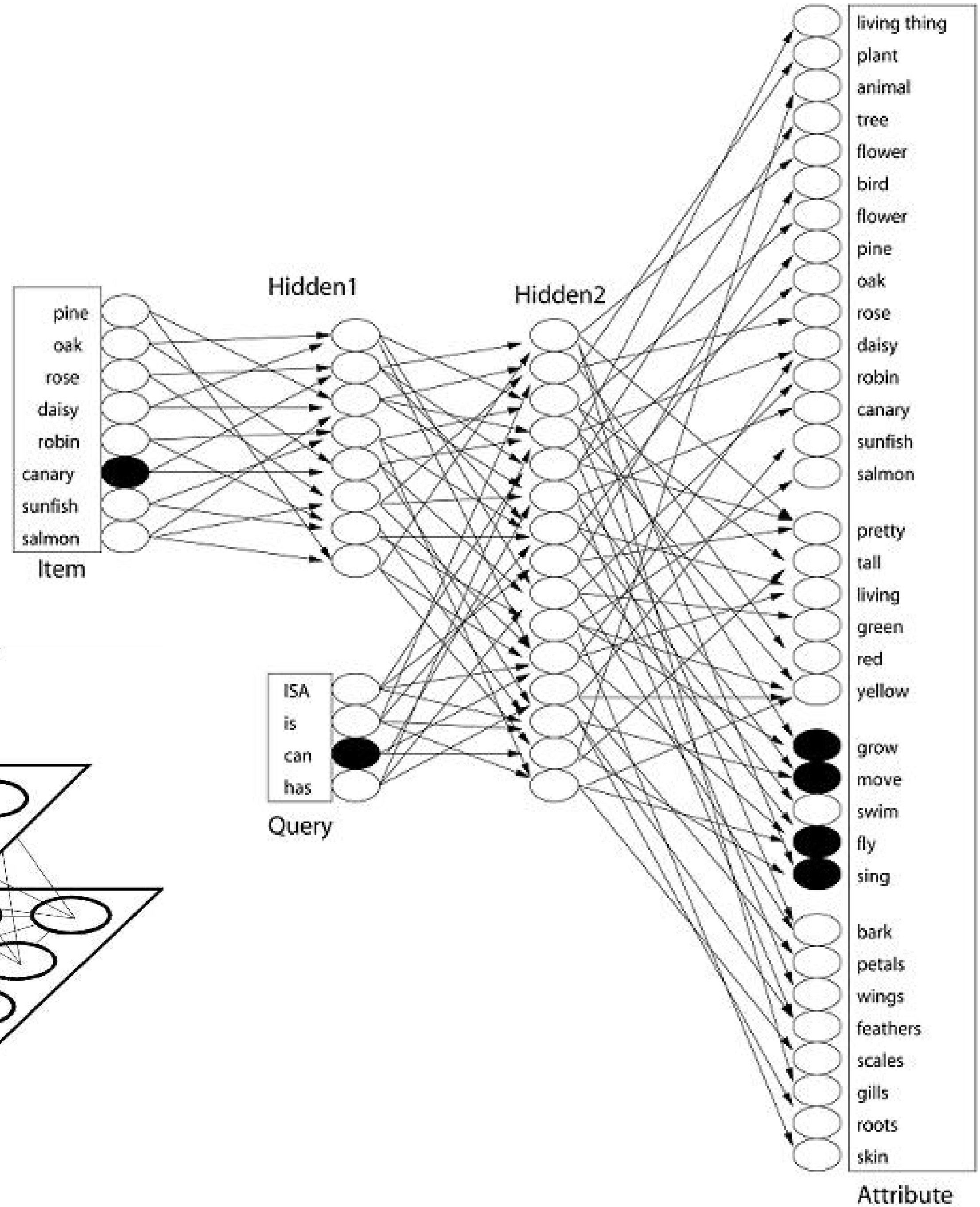
$$P(\underline{y} | \underline{x}, \underline{\theta}) = \prod_{i=1}^{n} \hat{y}_i^{\,y_i} (1-\hat{y}_i)^{1-y_i} = \prod_{i=1}^{n} P(y_i | x_i, \theta)$$

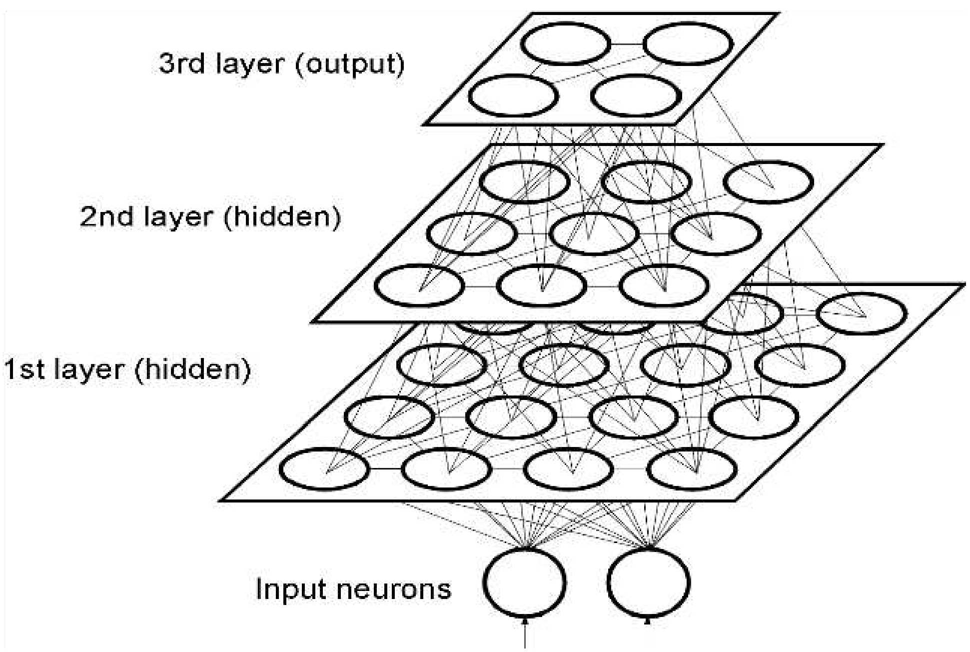
Cost:

$$c(\theta) = -\log P(\underline{y} | \underline{x}, \underline{\theta}) = -\sum_{i=1}^{n} y_i \log \hat{y}_i + (1-y_i) \log (1-\hat{y}_i)$$

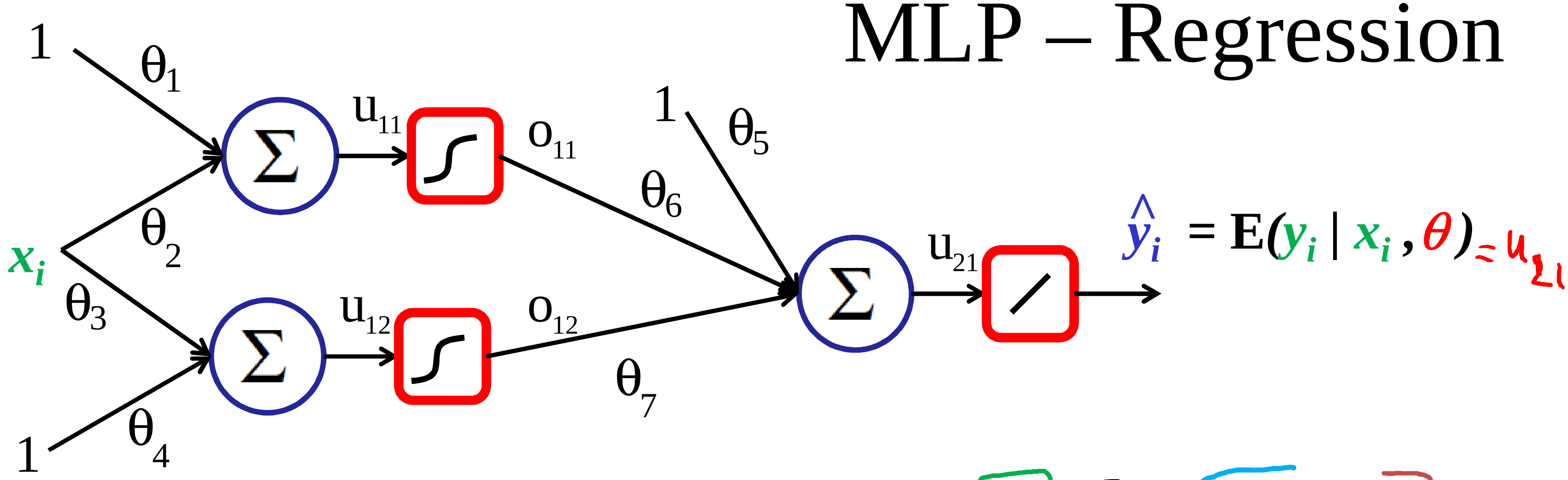
i.e minimize the cross-entropy error.


Age
Sex
Income
Occupation
Education
Social class
Geographic location
"Prospective" customer
"Non-prospective" customer


pine
oak
rose
daisy
robin
canary
sunfish
salmon
Item
Hidden1
Hidden2
ISA
is
can
has
Query
living thing
plant
animal
tree
flower
bird
flower
pine
oak
rose
daisy
robin
canary
sunfish
salmon
pretty
tall
living
green
red
yellow
grow
move
swim
fly
sing
bark
petals
wings
feathers
scales
gills
roots
skin
Attribute

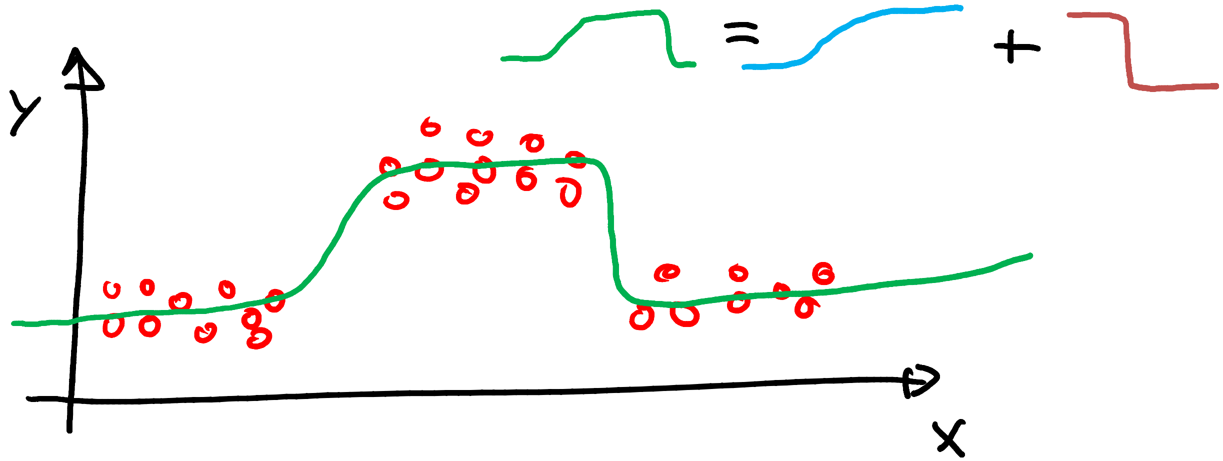

3rd layer (output)
2nd layer (hidden)
1st layer (hidden)
Input neurons

MLP – Regression

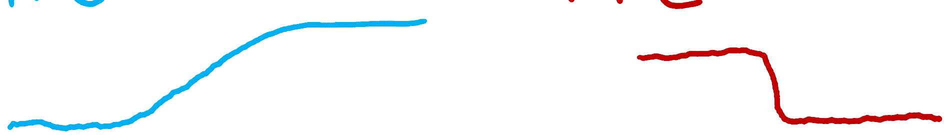


Data:

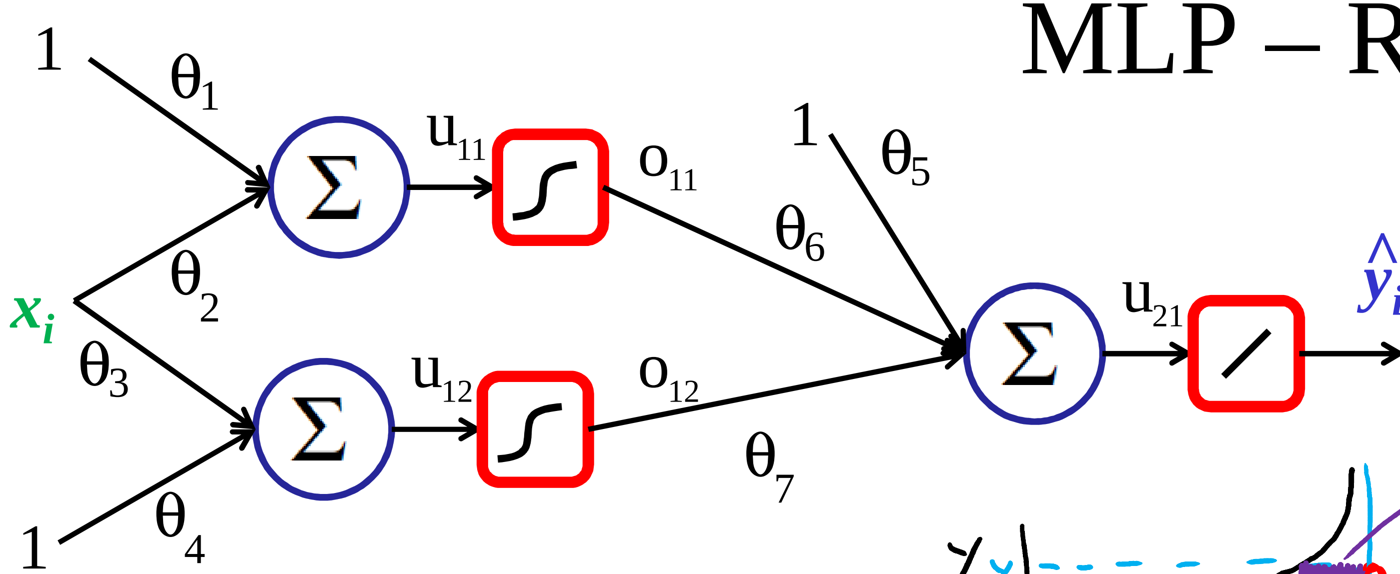
x_i	y_i
0.2	0.6
0.9	0.4
-0.6	5
-0.8	6.2



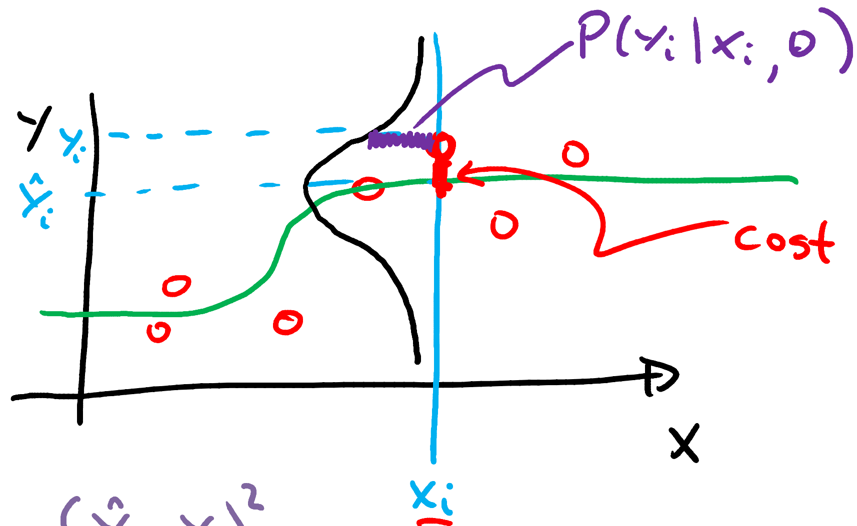
$$\hat{y}_i = \theta_5 + \frac{\theta_6}{1+e^{-\theta_1-\theta_2 x_i}} + \frac{\theta_7}{1+e^{-\theta_4-\theta_3 x_i}}$$



MLP – Regression



$\hat{y}_i = E(y_i \mid x_i, \theta) = u_{21}$

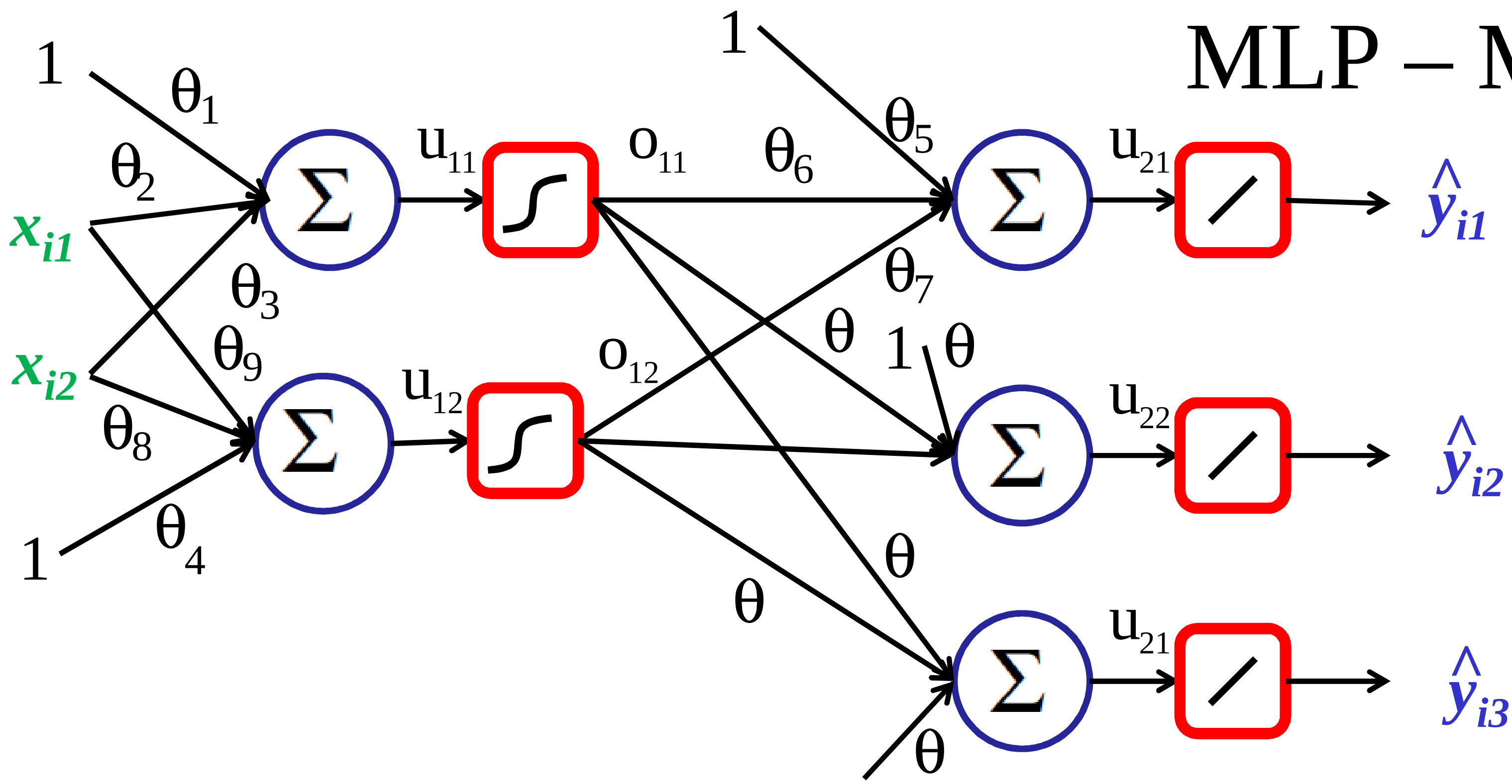


$$P(y_i \mid x_i, \theta) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(\hat{y}_i - y_i)^2}$$

$$c(\theta) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

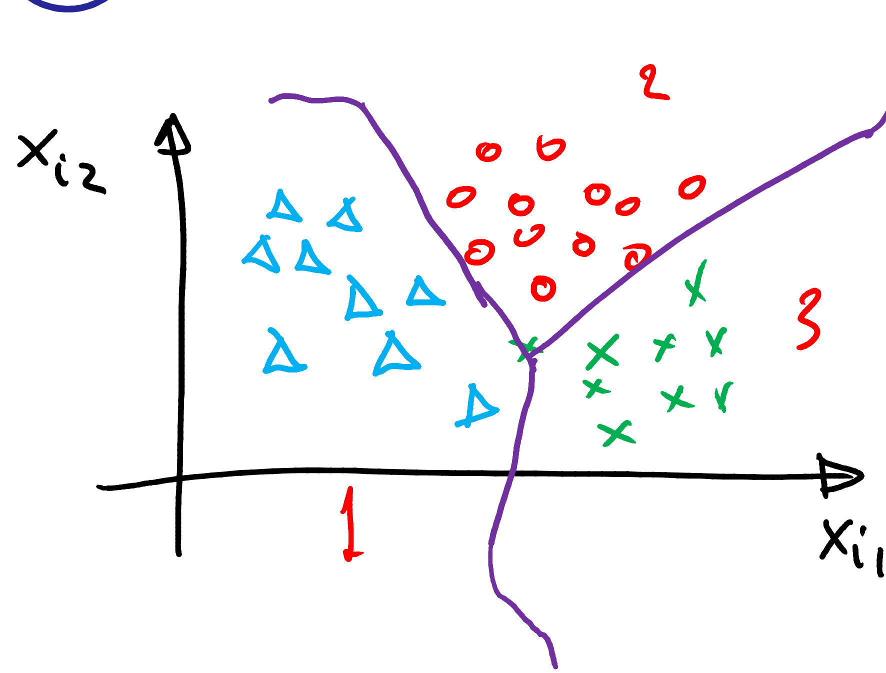
$$\hat{y}_i = f(\theta, x_i)$$

MLP – Multiclass

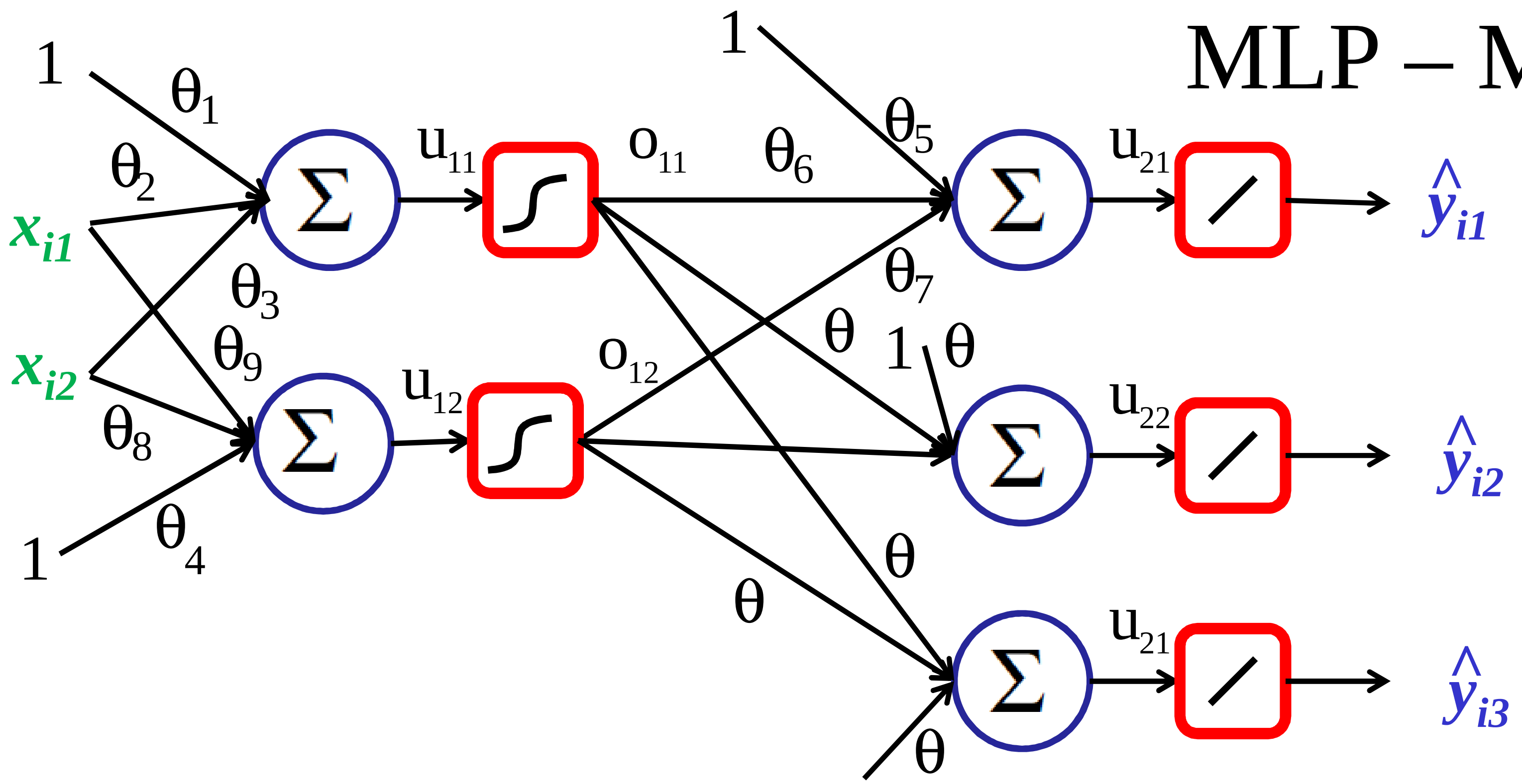


Data :

x_{i1}	x_{i2}	y_{i1}	y_{i2}	y_{i3}	
0.2	0.3	0	1	0	Class 2
-5	-6	1	0	0	Class 1
-20	4	1	0	0	" 1
42	6.8	0	0	1	Class 3



MLP – Multiclass



To get a probabilistic model, define: SOFTMAX

$$P(y_i = (0\,1\,0) \mid x_i, \theta) = P(y_i = 2 \mid x_i, \theta) = \frac{e^{\hat{y}_2}}{e^{\hat{y}_1} + e^{\hat{y}_2} + e^{\hat{y}_3}}$$

MLP – Multiclass

$$\mathbb{I}_2(y_i) = \begin{cases} 1 & y_i = 2 \\ 0 & \text{o.w.} \end{cases}$$

Then,

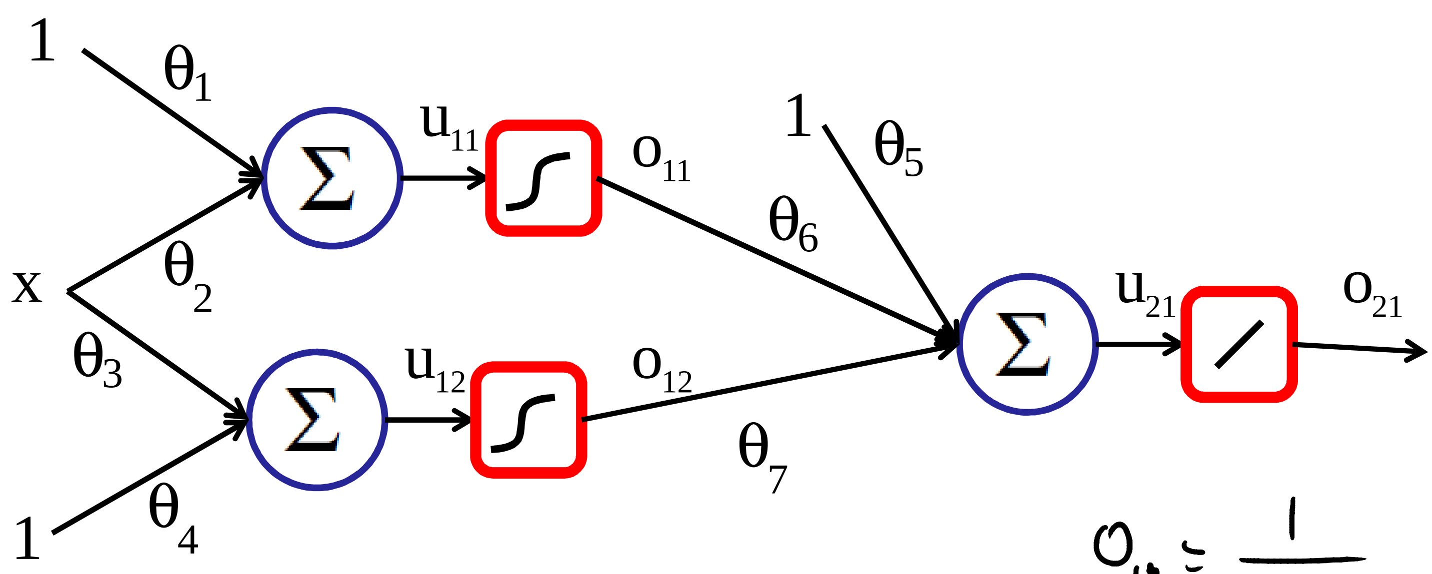
$$P(y_i \mid x_i, \theta) = \left[\frac{e^{\hat{y}_{i1}}}{\underbrace{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}_{\text{sum}}}\right]^{\mathbb{I}_1(y_i)} \left[\frac{e^{\hat{y}_{i2}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}\right]^{\mathbb{I}_2(y_i)} \left[\frac{e^{\hat{y}_{i3}}}{e^{\hat{y}_{i1}} + e^{\hat{y}_{i2}} + e^{\hat{y}_{i3}}}\right]^{\mathbb{I}_3(y_i)}$$

$$= \begin{cases} e^{\hat{y}_{i1}}/\text{sum} & y_i = 1 \\ e^{\hat{y}_{i2}}/\text{sum} & y_i = 2 \\ e^{\hat{y}_{i3}}/\text{sum} & y_i = 3 \end{cases}$$

Cost:

$$C(\theta) = -\log P(y \mid X, \theta) = -\sum_{i=1}^{n} \sum_{j=1}^{3} \mathbb{I}_j(y_i) \log \frac{e^{\hat{y}_{ij}}}{\text{sum}}$$

Backpropagation



$$O_{11} = \frac{1}{1+e^{-u_{11}}}$$

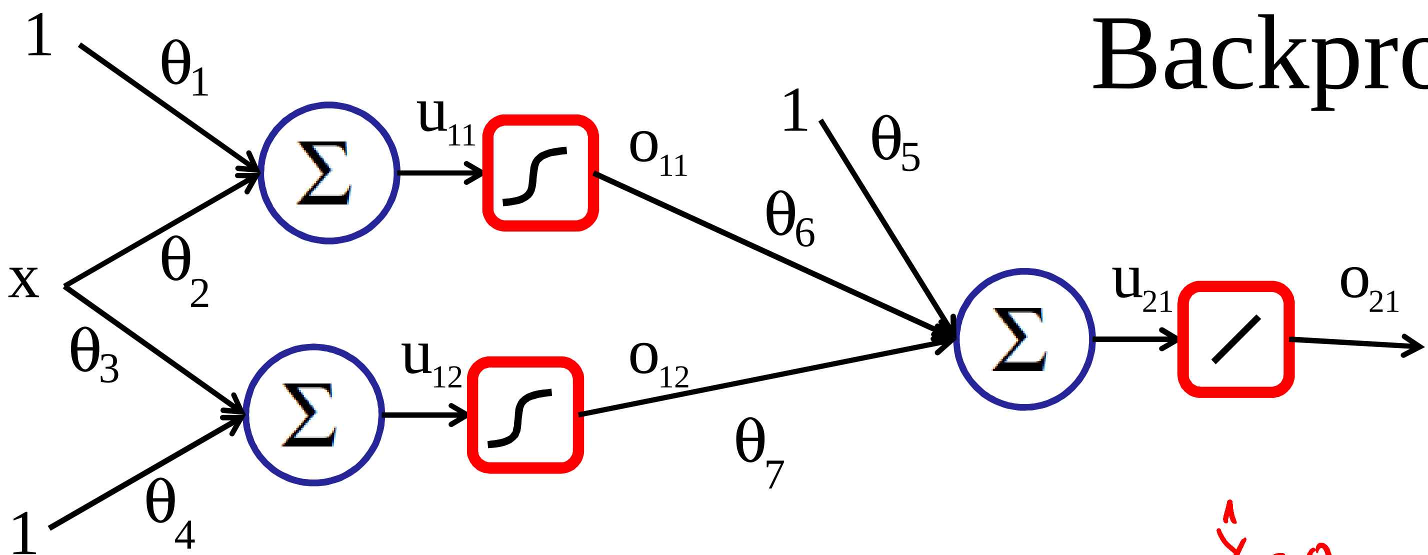
$$\hat{y} = O_{21} = u_{21} = \theta_5 + \theta_6 O_{11} + \theta_7 O_{12}$$

$$O_{12} = \frac{1}{1+e^{-u_{12}}}$$

$$u_{11} = \theta_1 + \theta_2 x$$

$$u_{12} = \theta_4 + \theta_3 x$$

Backpropagation



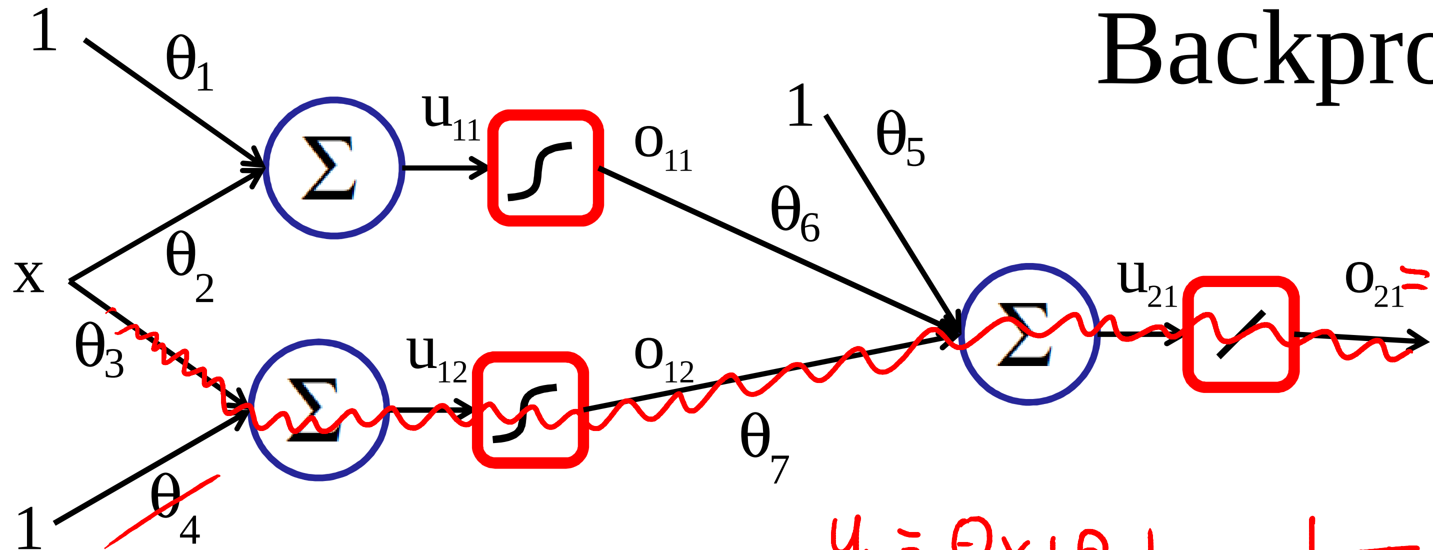
$$\hat{y} = o_{21} = \theta_5 + \theta_6 o_{11} + \theta_7 o_{12}$$

$$E(\theta) = \left(y_i - \hat{y}_i(x_i, \theta)\right)^2$$

$$\frac{\partial E(\theta)}{\partial \theta_j} = -2\left(y_i - \hat{y}_i(x_i, \theta)\right) \frac{\partial \hat{y}_i(x_i, \theta)}{\partial \theta_j}$$

$$\frac{\partial \hat{y}_i}{\partial \theta_5} = 1 \qquad \frac{\partial \hat{y}_i}{\partial \theta_6} = o_{11} \qquad \frac{\partial \hat{y}_i}{\partial \theta_7} = o_{12}$$

Backpropagation



$o_{21} = \hat{y} = u_{21}$

$\hat{y} = \theta_7 o_{12} + \theta_6 o_{11} + \theta_5$

$u_{12} = \theta_3 x + \theta_4 1$

$$\frac{\partial \hat{y}}{\partial \theta_3} = \frac{\partial \hat{y}}{\partial o_{12}} \frac{\partial o_{12}}{\partial u_{12}} \frac{\partial u_{12}}{\partial \theta_3}$$

$$= \theta_7 \, o_{12}[1-o_{12}] \, x$$

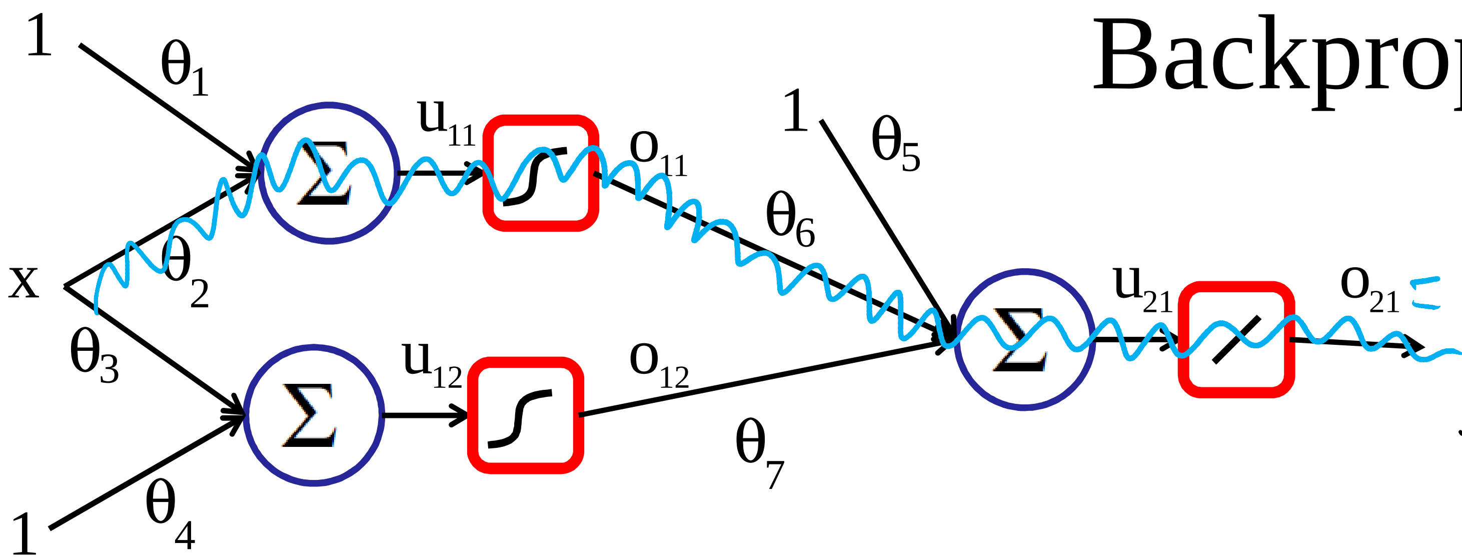
Thm

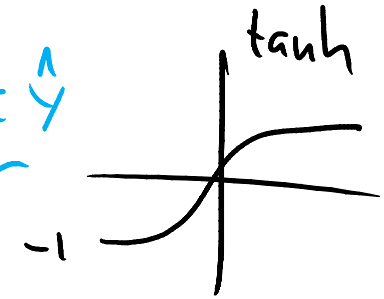
$$\frac{\partial o_{12}}{\partial u_{12}} = o_{12}(1-o_{12})$$

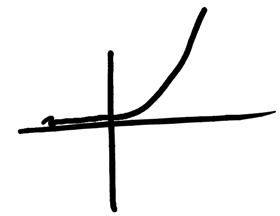
i.e.

$$\frac{\partial}{\partial x} \frac{1}{1+e^{-x}} = \left(\frac{1}{1+e^{-x}}\right)\left(1-\frac{1}{1+e^{-x}}\right)$$

Backpropagation







Batch:

iteration

$$\Theta_j^{(k+1)} = \Theta_j^{(k)} + \sum_{i=1}^{n} (y_i - \hat{y}_i) \frac{\partial \hat{y}_i}{\partial \Theta_j}$$

Online:
e.g.

$$\Theta_2^{i} = \Theta_2^{i-1} + \eta (y_i - \hat{y}_i)\, \Theta_6^{i-1} (o_{11}^i (1 - o_{11}^i)) x_i$$

Next lecture

In the next lecture, we will continue working with neural networks.