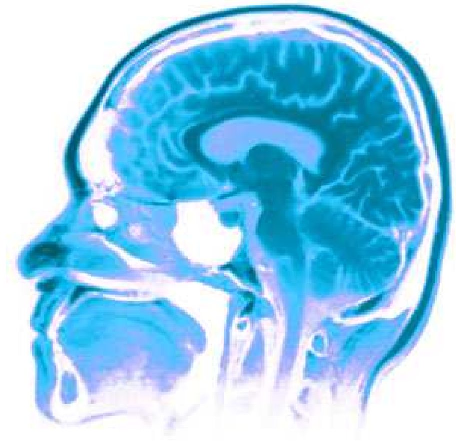
CPSC340



Inference in Probabilistic Graphical Models

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University of British Columbia

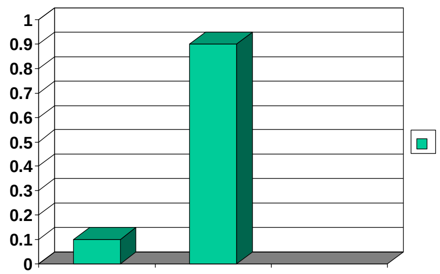
Outline of the lecture

This lecture is devoted to the problem of inference in probabilistic graphical models (aka Bayesian nets). The goal is for you to:

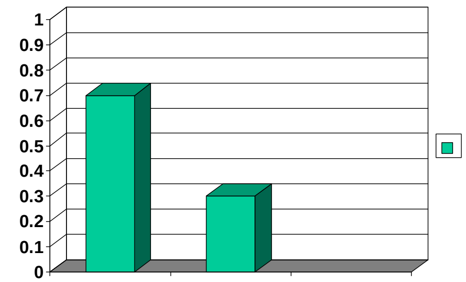
- Practice marginalization and conditioning.
- Learn how to apply dynamic programming to do inference.

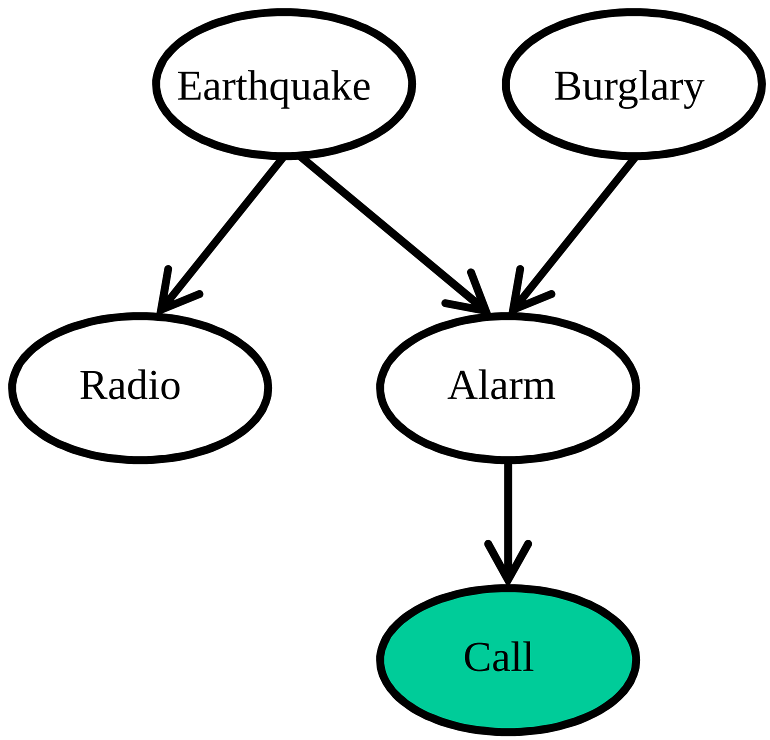
Inference

$P(E=t|C=t)=0.1$



$P(B=t|C=t) = 0.7$

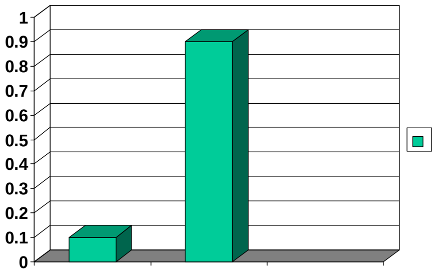




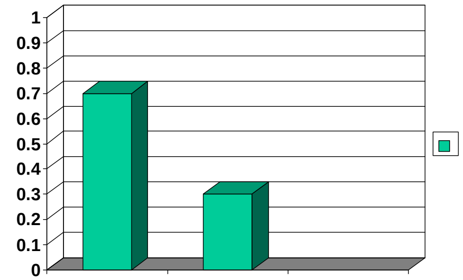
C=t

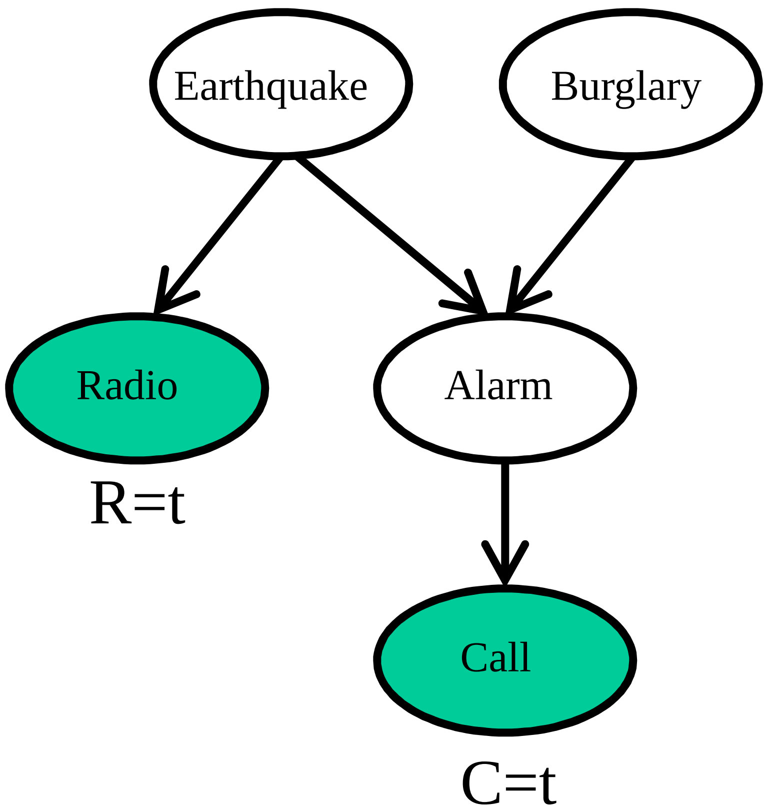
Inference

P(E=t|C=t)=0.1



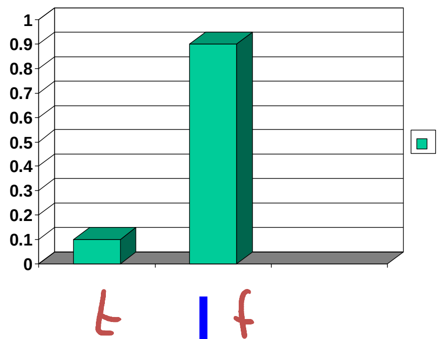
P(B=t|C=t) = 0.7



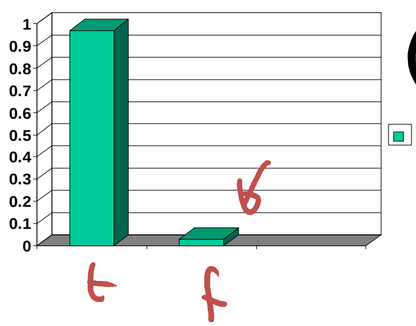


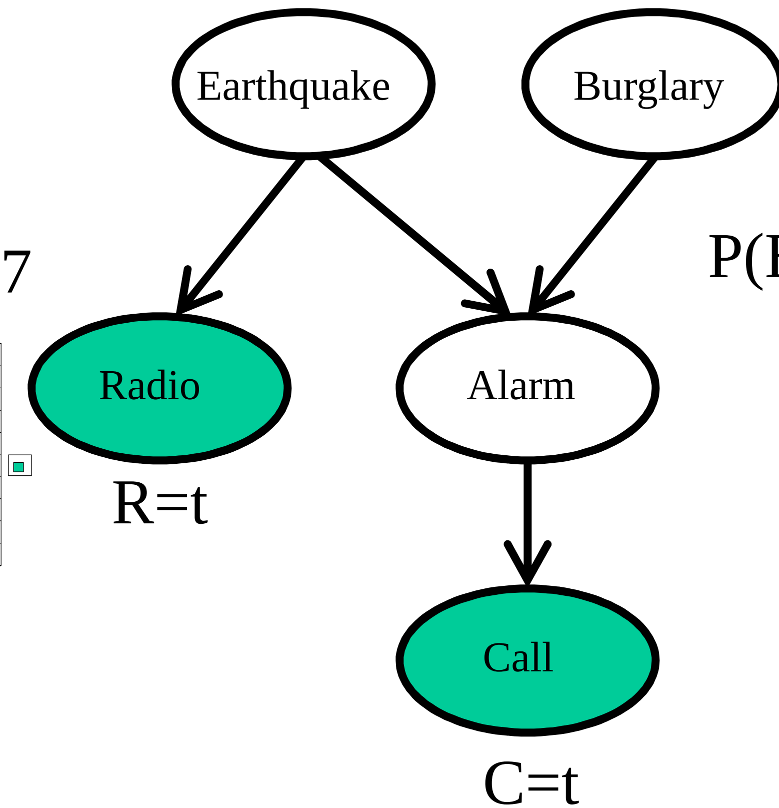
Inference

P(E=t|C=t)=0.1

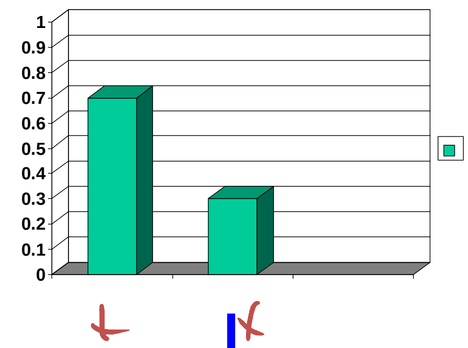


P(E=t|C=t,R=t)=0.97

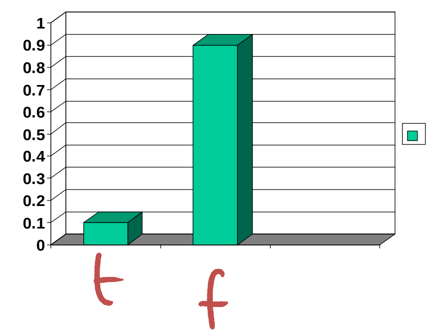




P(B=t|C=t) = 0.7



P(B=t|C=t,R=t) = 0.1



Explaining away effect

The sprinkler network

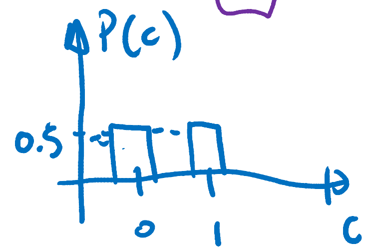
$$P(c) = \begin{array}{c|c} 0 & 1 \\ \hline 0.5 & 0.5 \end{array} \qquad P(c=0) = 0.5$$

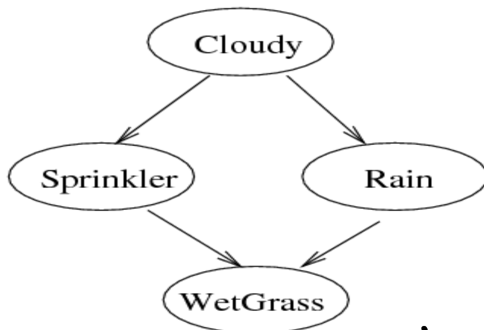
$$P(s|c=1) = \begin{array}{c|c} 0 & 1 \\ \hline 0.9 & 0.1 \end{array}$$

$$\sum_s P(s|c=1) = 1$$

P(c)

P(C=F)	P(C=T)
0.5	0.5



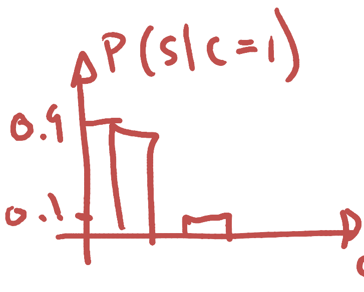


P(s|c)

C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

$$P(c=0, s=0) = P(s=0|c=0)P(c=0)$$

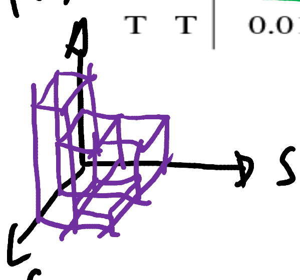


S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

$$P(w|s=0, r=1) = \begin{array}{c|c} 0 & 1 \\ \hline 0.1 & 0.9 \end{array}$$

$$P(c,s) = \begin{array}{c|cc} & s=0 & s=1 \\ \hline c=0 & .25 & .25 \\ c=1 & 0.45 & 0.05 \end{array}$$

p(c,s)



$$p(C, S, R, W) = p(C)p(S|C)p(R|C)p(W|S, R)$$

Inference in DAGs

Let us use 0 to denote false and 1 to denote true.

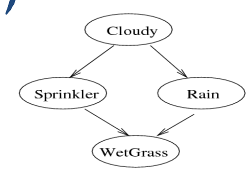
What is the ***marginal probability****,* ***P(S=1)****, that the sprinkler is on?*

$$P(S=1) = \sum_{C=0}^{1}\sum_{R=0}^{1}\sum_{W=0}^{1} P(C,R,W,S=1)$$

$$= \sum_{C}\sum_{R}\sum_{W} P(C)P(S=1|C)P(R|C)P(W|S=1,R)$$

$P(C)$

P(C=F)	P(C=T)
0.5	0.5



$P(S|C)$

C	P(S=F)	P(S=T)
F	0.5	0.5
T	0.9	0.1

$P(R|C)$

C	P(R=F)	P(R=T)
F	0.8	0.2
T	0.2	0.8

S	R	P(W=F)	P(W=T)
F	F	1.0	0.0
T	F	0.1	0.9
F	T	0.1	0.9
T	T	0.01	0.99

$P(W|R,S)$

Brute force (exponential) approach

*What is the **marginal probability**, **P(S=1)**, that the sprinkler is on?*

$$\sum_{C}\sum_{R}\sum_{W=0}^{1} P(W|S=1,R)\,P(S=1|C)\,P(R|C)\,P(C)$$

$ab + ac$

$a(b+c)$

$$= \overset{000}{} \; P(W=0|S=1,R=0) \times P(S=1|C=0) \times P(R=0|C=0) \times P(C=0)$$

$$ \;\; 0.1 \qquad\qquad 0.5 \qquad\qquad 0.8 \qquad\qquad 0.5$$

$$\overset{001}{+} \; P(W=1|S=1,R=0)\,P(S=1|C=0)\,P(R=0|C=0)\,P(C=0)$$

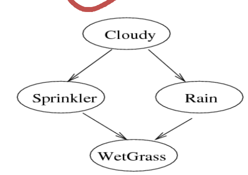
$$\overset{010}{+} \; P(W=0|S=1,R=1)\,P(S=1|C=0)\,P(R=1|C=0)\,P(C=0)$$

$$\overset{011}{+} \; \cdots$$

$$\vdots$$

$$\overset{111}{+}$$

P(C=F)	P(C=T)
0.5	0.5



C	P(S=F)	P(S=T)
F	0.5	0.5
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S R	P(W=F)	P(W=T)
F F	1.0	0.0
T F (10)	0.1	0.9
F T	0.1	0.9
T T	0.01	0.99

Brute force (exponential) approach

*What is the **marginal probability**, **P(S=1)**, that the sprinkler is on?*

```
PROD
FOR R=0:1:1
    FOR C=0:1:1
        FOR W=0:1:1
            PROD = PROD + P(C)P(R|C)P(S=1|C)P(W|S=1,R)
        END
    END
END
```

Smart approach: variable elimination, *aka* dynamic programming, *aka* distributive law

What is the ***marginal probability***, ***P(S=1)***, *that the sprinkler is on?*

$$P(S=1) = \sum_C \sum_R \sum_W P(W|S=1,R)\, P(R|C)\, P(C)\, P(S=1|C)$$

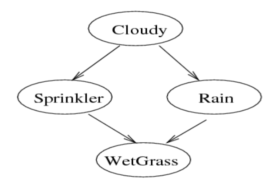
$$= \sum_C \sum_R P(R|C)\, P(C)\, P(S=1|C) \boxed{\cancelto{1}{\sum_W P(W|S=1,R)}} \sim \phi = 1$$

$$= \sum_C P(C)\, P(S=1|C) \boxed{\cancelto{1}{\sum_R P(R|C)}} \sim \psi = 1$$

$$= P(S=1|C=0)\, P(C=0) + P(S=1|C=1)\, P(C=1)$$

$$= 0.3$$

P(C=F)	P(C=T)
0.5	0.5



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F	F	1.0	0.0
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T	T	0.01	0.99

Smart approach: variable elimination, *aka* dynamic programming, *aka* distributive law

What is the ***marginal probability****,* ***P(S=1)****, that the sprinkler is on?*

$\psi = 0$

$\phi = 0$

$\Theta = 0$

FOR $W = 0:1:1$

$\quad \phi_R = \phi_R + P(W \mid S=1, R)$

END

FOR $R = 0:1:1$

$\quad \psi_C = \psi_C + P(R \mid C)\,\phi_R$

END

FOR $C = 0:1:1$

$\quad \Theta = \Theta + P(S=1 \mid R)\,P(C)\,\psi_C \longrightarrow \Theta = 0.3$

END

Smart approach: variable elimination, *aka* dynamic programming, *aka* distributive law

We won't implement the general code in this course. To do this one needs to learn about the ***junction tree*** *data structure. This structure, once created, enables us to conduct any query on the graph very efficiently.*

These exact algorithms work well for small graphs and for graphs that are ***trees*** *or close to trees (have low tree-width). For large densely connected graphs we require the use of algorithms beyond the scope of this course. One of those algorithms is called* ***Gibbs sampling***.

In the next class, we will address a tree of great interest: a chain of nodes. For discrete models this model is known by the name ***HMM***.

Inference in DAGs

What is the ***posterior probability****,* ***P(S=1|W=1),*** *that the sprinkler is on given that the grass is wet?*

$$P(S=1 \mid W=1) = \frac{P(S=1, W=1)}{P(W=1)}$$

$$P(W=1) = \sum_{S} \sum_{C} \sum_{R} P(S, W=1, C, R)$$

$$P(S=1, W=1) = \sum_{C} \sum_{R} P(S=1, W=1, C, R)$$

Inference in DAGs

What is the ***posterior probability***, ***P(S=1|W=1,R=1)***, *that the sprinkler is on given that the grass is wet and it is raining?*

$$P(S=1|W=1,R=1) = P(S=1 \mid (W=1,R=1))$$

$$= \frac{P(S=1,W=1,R=1)}{P(W=1,R=1)}$$

Next lecture

In the next lecture, we will learn to do inference on a tree-structured graphical model, known as hidden Markov model (HMM)