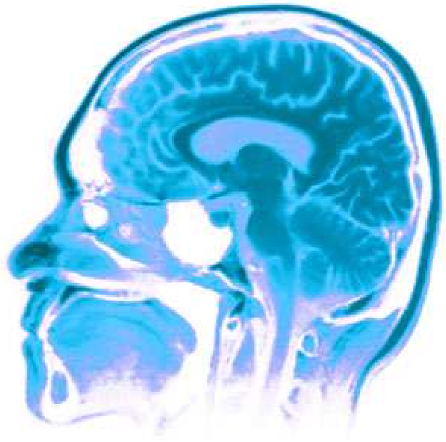
CPSC340



Linear algebra revision

Nando de Freitas

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University of British Columbia

Outline of the lecture

This lecture revises linear algebra. You are expected to become familiar with:

- Matrix multiplication
- Eigen value decompositions
- Basic linear algebra definitions: Transpose, orthogonal matrix, diagonal matrix, symmetric matrix, vector norm, etc.

Matrix Multiplication

$$\boxed{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}$$

$$y_1 = A_{11}x_1 + A_{12}x_2$$

$$y_2 = A_{21}x_1 + A_{22}x_2$$

$$\boxed{\underline{\underline{A}}\,\underline{x} = \underline{y}}$$

$$y_i = A_{i1}x_1 + A_{i2}x_2$$

$$\boxed{y_i = \sum_j A_{ij}x_j}$$

$$\underline{\underline{A}} = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

$$\underline{a}_1 = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}$$

$$\boxed{\underline{\underline{A}}\,\underline{x} = x_1\underline{a}_1 + x_2\underline{a}_2} = \underline{y}$$

$$\begin{bmatrix} \alpha & (1-\alpha) \end{bmatrix} \begin{bmatrix} \theta & 1-\theta \\ \beta & 1-\beta \end{bmatrix} = \underline{Y}^T = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

$$\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad \underline{Y}^T = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$$

<u>Lemma</u>

If $\underline{X}^T \underline{\underline{G}} = Y^T$

and $\sum_i X_i = 1$ and $\sum_j G_{ij} = 1$ for all i, Then

$\sum_j Y_j = 1$

$$Y_j = \sum_i X_i G_{ij} \Rightarrow \sum_j Y_j = \sum_j \sum_i X_i G_{ij}$$

$$= \sum_i X_i \sum_j G_{ij} = \sum_i X_i = 1$$

$$\underline{\underline{A}} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$

$$\underline{\underline{A}}\,\underline{x} - \lambda \underline{\underline{I}}\,\underline{x} = \underline{0}$$

$$\left[\underline{\underline{A}} - \lambda \underline{\underline{I}}\right] \underline{x} = \underline{0}$$

$$\begin{bmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x}$$ right (of A) eigenvectors

$$\underline{x}^T \underline{\underline{A}} = \lambda \underline{x}^T$$ left (of A) eigenvectors

eigenvalues

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

identity

$$\begin{vmatrix} 1-\alpha-\lambda & \alpha \\ \beta & 1-\beta-\lambda \end{vmatrix} = (1-\alpha-\lambda)(1-\beta-\lambda) - \alpha\beta = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 1-\alpha-\beta$$

$$\underline{\underline{A}}\,\underline{\underline{x}} = 1\,\underline{\underline{x}} \quad \Rightarrow \quad (\underline{\underline{A}} - 1\underline{\underline{I}})\,\underline{x} = \underline{0}$$

$$\begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underline{1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

$$-\alpha x_1 + \beta x_2 = 0$$

$$\alpha x_1 - \beta x_2 = 0$$

$$\underline{x} = \begin{bmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{bmatrix} \qquad \sum_i x_i = 1$$

$$\Pi_0^T G = \Pi_1^T$$

$$\Pi_1^T G = \Pi_2^T$$

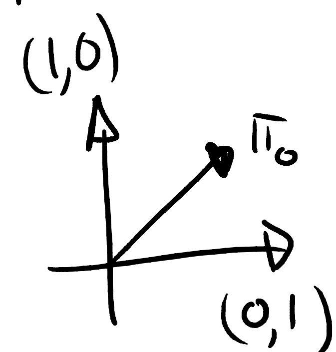
$$\vdots$$

$$\Pi_{m-1}^T G = \Pi_m^T$$

$$\Pi_0^T G^m = \Pi_m^T$$

Let $x_i^T G = \lambda_i x_i^T$, $\lambda_1 > \lambda_2 > \lambda_3 > \dots$, $c_i > 0$

Assume G (an $N \times N$ matrix) has N distinct eigenvalues. Suppose the x_i form a bases



$$\Pi_0^T = \sum_{i=1}^{N} c_i x_i^T$$

Then,

$$\Pi_0^T G = \sum_{i=1}^{N} c_i x_i^T G = \sum_{i=1}^{N} c_i \lambda_i x_i^T = \Pi_1^T$$

$$\Pi_0^T G^2 = \Pi_1^T G = \sum_{i=1}^{N} c_i \lambda_i \underbrace{x_i^T G}_{\lambda_i x_i^T} = \sum_{i=1}^{N} c_i \lambda_i^2 x_i^T = \Pi_3^T$$

$$\vdots$$

$$\Pi_0^T G^m \approx \sum_{i=1}^{N} c_i \lambda_i^m x_i^T \approx c_1 \lambda_1^m x_1^T + c_2 \lambda_2^m x_2^T + \dots$$

$$\underline{\Pi}_0 \underline{\underline{G}}^m \approx \underbrace{\overbrace{\left(c_1 \lambda_1^m\right)}^{\text{const}} \underline{X}_1^T}_{\Pi_m}$$ as m is large.

$$\sum_i \Pi_{m,i} = 1$$

$\underline{\Pi}$ is the normalized eigenvector $\underline{X}_1^T$

$$\underline{\underline{A}}\,\overbrace{\begin{bmatrix}\underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_m\end{bmatrix}}^{\underline{\underline{X}}} = \begin{bmatrix}\boxed{\lambda_1 \underline{x}_1} & \boxed{\lambda_2 \underline{x}_2} & \cdots & \lambda_m \underline{x}_m\end{bmatrix}$$

$$\begin{bmatrix}\boxed{\underline{\underline{A}}\,\underline{x}_1} & \boxed{\underline{\underline{A}}\,\underline{x}_2} & \cdots & \underline{\underline{A}}\,\underline{x}_m\end{bmatrix} = \begin{bmatrix}\underline{x}_1 & \underline{x}_2 & \cdots & \underline{x}_m\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0 & 0 & \cdots & 0\\ 0 & \lambda_2 & & & & \\ \vdots & & \ddots & & & \\ 0 & \cdots & & & & \lambda_m\end{bmatrix}$$

$$\boxed{\underline{\underline{A}}\,\underline{x}_i = \lambda_i \underline{x}_i \quad \forall i}$$

$$\underline{\underline{A}}\,\underline{\underline{X}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}} \quad \Rightarrow \quad \boxed{\underline{\underline{A}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^{-1}}$$

$$\underline{\underline{A}}\,\underline{\underline{X}}\,\underline{\underline{X}}^{-1} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^{-1}$$

If $\underline{\underline{A}} = \underline{\underline{A}}^T$ ie. $\underline{\underline{A}}$ is Symmetric

$$\underline{\underline{A}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^T$$

$$\boxed{\underline{\underline{X}}^{-1} = \underline{\underline{X}}^T}$$

ie.

$\underline{\underline{X}}$ is Orthogonal

$$\underline{\underline{X}} = [\underline{x}_1 \ \ \underline{x}_2 \ \cdots \ \underline{x}_m] \qquad \underline{x}_i^T \underline{x}_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

orthonormal.

$$\underline{\underline{A}} = \underline{\underline{X}}\,\underline{\underline{\Lambda}}\,\underline{\underline{X}}^T$$

if $A = A^T$ (symmetric)

$A \in \mathbb{R}^{m \times m}$

i.e. A is mxm and has real numbers.

$$\underline{\underline{A}} = \begin{bmatrix} \underline{x}_1 & \cdots & \underline{x}_m \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \\ \vdots \\ x_m^T \end{bmatrix}$$

$$= \sum_{i=1}^{m} \underline{x}_i \lambda_i \underline{x}_i^T = \lambda_1 \underline{x}_1 \underline{x}_1^T + \lambda_2 \underline{x}_2 \underline{x}_2^T + \cdots$$

Vector norm

$\underline{x} \in \mathbb{R}^{m \times 1}$

$$\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots x_m^2} = \sqrt{\underline{x}^T \underline{x}}$$ (Length of vector)

if $\underline{\underline{Q}}$ is an $m \times m$ orthogonal matrix : $(Q^T = Q^{-1})$:

$$\|Qx\|_2 = \sqrt{(Qx)^T(Qx)} = \sqrt{x^T Q^T Q x}$$

$$= \sqrt{x^T x} = \|x\|_2$$

- Q rotates x, but does not change its length.

Next lecture

In the next lecture, we introduce the SVD.