Lecture 5 - *Linear Supervised Learning*

OBJECTIVE: Linear regression is a supervised learning task. It is of great interest because:

- Many real processes can be approximated with linear models.
- Linear regression appears as part of larger problems.
- It can be solved analytically.
- It illustrates many of the approaches to machine learning.

Textbook: Pages 41–49.

Given the data $\{x_{1:n}, y_{1:n}\}$, with $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$, we want to fit a hyper-plane that maps x to y.

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Mathematically, the linear model is expressed as follows:

$$y_i = \theta_0 + \sum_{j=1}^{d} x_{ij}\theta_j$$

We let $x_{i,0} = 1$ to obtain

$$y_i = \sum_{j=0}^{d} x_{ij}\theta_j$$

In matrix form, this expression is

$$Y = X\theta$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_{10} & \cdots & x_{1d} \\ \vdots & \vdots & \vdots \\ x_{n0} & \cdots & x_{nd} \end{bmatrix} \begin{bmatrix} \theta_0 \\ \vdots \\ \theta_d \end{bmatrix}$$

If we have several outputs $y_i \in \mathbb{R}^c$, our linear regression expression becomes:

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We will present several approaches for computing θ.

◇ OPTIMIZATION APPROACH

Our aim is to mininimise the quadratic cost between the output labels and the model predictions

$$C(\theta) = (Y - X\theta)^T (Y - X\theta)$$

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We will need the following result from matrix differentiation: $\frac{\partial A}{\partial \theta} = A^T$.

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$$\frac{\partial C}{\partial \theta} =$$

These are the **normal equations**. The solution (estimate) is:

$$\widehat{\theta} =$$

The corresponding predictions are

$$\widehat{Y} = HY =$$

where H is the "hat" matrix.

◇ GEOMETRIC APPROACH

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$$X^T(Y - \widehat{Y}) =$$

◇ PROBABILISTIC APPROACH

Univariate Gaussian Distribution

The probability density function of a Gaussian distribution is given by

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}.$$

where μ is the mean or center of mass and σ^2 is the variance.

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Our short notation for Gaussian variables is $X \sim \mathcal{N}(\mu, \sigma^2)$.

Multivariate Gaussian Distribution

Let $x \in \mathbb{R}^n$. The pdf of an n-dimensional Gaussian is given by

$$p(x) = \frac{1}{2\pi^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

where

$$\mu = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \mathbb{E}(x_1) \\ \vdots \\ \mathbb{E}(x_n) \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} \sigma_{11} \cdots \sigma_{1n} \\ \cdots \\ \sigma_{n1} \cdots \sigma_{nn} \end{pmatrix} = \mathbb{E}[(X-\mu)(X-\mu)^T]$$

with $\sigma_{ij} = \mathbb{E}[X_i - \mu_i)(X_j - \mu_j)^T]$.

We can interpret each component of x, for example, as a feature of an image such as colour or texture. The term $\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$ is called the **Mahalanobis distance**. Conceptually, it measures the distance between x and μ.

Maximum Likelihood

If our errors are Gaussian distributed, we can use the model

$$Y = X\theta + \mathcal{N}(0, \sigma^2 I)$$

Note that the mean of Y is $X\theta$ and that its variance is $\sigma^2 I$. So we can equivalently write this expression using the probability density of Y **given** X, θ and σ:

$$p(Y|X,\theta,\sigma) = \left(2\pi\sigma^2\right)^{-n/2} e^{-\frac{1}{2\sigma^2}(Y-X\theta)^T(Y-X\theta)}$$

The maximum likelihood (ML) estimate of θ is obtained by taking the derivative of the log-likelihood, $\log p(Y|X,\theta,\sigma)$. The idea of maximum likelihood learning is to maximise the likelihood of seeing some data Y by modifying the parameters (θ, σ).

The ML estimate of θ is:

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Proceeding in the same way, the ML estimate of σ is:

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