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A Smooth Dynamical System that Counts in Binary

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Abstract—**This paper presents a smooth dynamical system that implements a toggle flip-flop. The flip-flop is described as a system of smooth, non-linear ODE's. We identify a period-2, invariant set of this system, and show that this corresponds to the discrete state transitions of a discrete model. We show that this behaviour is robust for a large class of inputs and that these toggle elements can be composed to implement a binary counter of any number of bits.**

I. Introduction

High performance digital designs often require verifying that a circuit as modeled by differential equations (e.g. SPICE) implements a desired discrete operation. This paper presents a simple dynamical system that implements the behaviour of a toggle flip-flop. We use concepts from dynamical systems theory to show that the ordinary differential equation (ODE) model implements a discrete toggle. The simple model that we present is mathematically tractable and independent of any particular fabrication technology. In [1], [2] similar methods were employed to verify the CMOS toggle from [3].

Our toggle is modeled by a system of non-linear differential equations. The input to the toggle alternates between high and low values, and we use the Brockett's annulus construction [4] to specify ranges for the high and low values of the signal as well as ranges for the rise and fall times. We show that for all such inputs, there is an invariant set with period twice that of the input signal that contains all possible trajectories. Thus, the output alternates between high and low values at half of the rate of the input.

In [4], Brockett presented smooth dynamical systems that count and perform various other arithmetic and logical operations. Brockett's counter represented the value of the count by a variable whose value is proportional to the count and therefore increases without bound. Our binary counter construction more closely resembles the operation of real digital circuits. Our toggle element has bounded inputs and outputs, and we show that the output of our toggle element satisfies the constraints for the input. Thus, our toggle elements can be coupled in a chain to implement a binary ripple counter. In dynamical systems terminology, our system scales: by analysing a simple toggle element, we can verify properties of a system of much higher dimension. As in Brockett's examples, our ODE model for the toggle is infinitely differentiable.

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II. The Toggle Element

We use a simple model where values near $+1$ represent a logical true and values near -1 represent a logical false. Our toggle is described by the ODE:

$$\begin{aligned} \dot{x} &= (1+\theta)(x-x^3) + (1-\theta)(-y-x) \\ \dot{y} &= (1+\theta)(x-y) + (1-\theta)(y-y^3) \end{aligned} \tag{1}$$

where θ is the input (i.e. "clock") of the toggle and x and y are the outputs. Toggle behaviour requires that x and y change at half the rate of θ. As shown below, x changes on falling edges on θ and y changes on rising edges. This gives rise to the desired toggle behaviour.

First, consider the case when $\theta = +1$. The model simplifies to

$$\begin{aligned} \dot{x} &= 2(x-x^3) \\ \dot{y} &= 2(x-y) \end{aligned} \tag{2}$$

The system has stable equilibria at $(x,y) = (1,1)$ and $(x,y) = (-1,-1)$, and a saddle point at $(x,y) = (0,0)$. The y-axis is the separator between the basins of attraction for the two stable equilibria. Likewise, when $\theta = -1$ the system has stable equilibria at $(x,y) = (1,-1)$ and $(x,y) = (-1,1)$ with basins of attraction separated by the x-axis, and there is a saddle point at $(x,y) = (0,0)$.

This attractor structure gives rise to the toggle behaviour of the system. Consider an instant when $\theta = 1$ and (x,y) is near $(1,1)$. If θ then transitions to -1, the point $(x,y) = (1,1)$ is in the basin of attraction of $(-1,1)$ and will asymptotically approach that point. Similar arguments for further transitions of θ suggest that trajectories traverse the cycle depicted in figure 1. The points marked with $\oplus$ and $\ominus$ are the attractors when $\theta = +1$ and $\theta = -1$ respectively, and the saddle point is marked with $\otimes$. The trajectory drawn shows the period-2 attractor when $\theta = \sin(t)$. Figure 2 shows the same trajectory plotted as functions of time. The requirement that the output of the toggle must change at half the rate of the input is reflected by the period-2 attractor of the continuous system.

Because we are interested in smooth models, we must consider the behaviour of the toggle when θ is between $+1$ and -1. The equilibrium points for the toggle can be determined by solving the polynomial equations for $\dot{x} = \dot{y} = 0$. As shown in figure 3, the stable attractors move from $(1,1)$ and $(-1,-1)$ to the origin as θ decreases from $+1$ to $1/\sqrt{5}$. Likewise, as θ continues to decrease from $-1/\sqrt{5}$ to -1, the stable attractors move away from the origin out to $(1,-1)$ and $(-1,1)$. There are Hopf-bifurcations at $\theta = \pm 1/\sqrt{5}$.

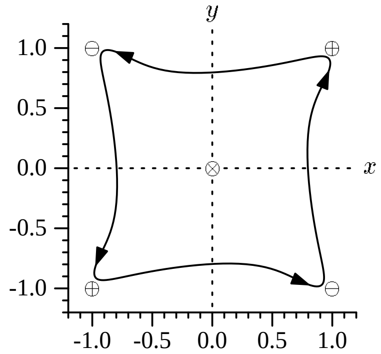

Fig. 1. Phase portrait of the toggle

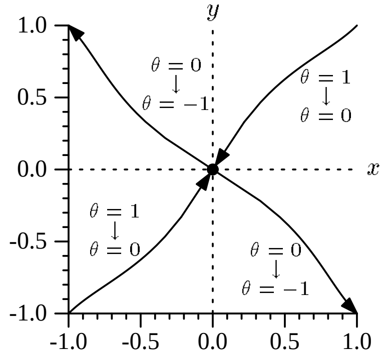

Fig. 3. Locii of attractors as θ changes

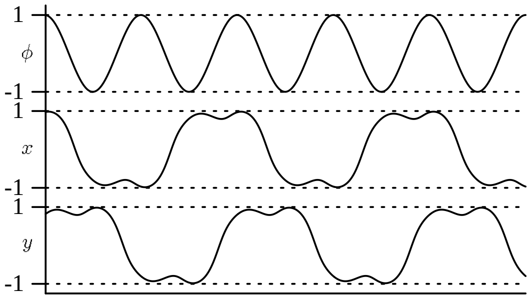

Fig. 2. Time waveforms of the toggle

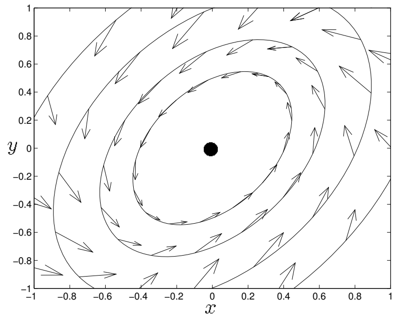

Fig. 4. Flow when $\theta = 0.2$

For $-1/\sqrt{5} < \theta < 1/\sqrt{5}$ all trajectories spiral toward the origin. This can be seen by considering a "linearized" version of the system:

$$\begin{aligned} \dot{x} &= (1+\theta)x + (1-\theta)(-y-x) \\ \dot{y} &= (1+\theta)(x-y) + (1-\theta)y \end{aligned} \quad (3)$$

Solutions to the linearized system are ellipses with axes along $y = x(\sqrt{5}+1)/2$ and $y = -x(\sqrt{5}-1)/2$. The flows of the original system are always inward with respect to these ellipses as shown in figure 4. This can be established formally by showing that the derivative of the radius in the elliptical coordinate system is always negative.

Toggle operation requires that θ be a suitable function of time. The analysis for fixed values of θ provides an intuitive model for a toggle with changing θ. When θ is close to $+1$, trajectories will approach either $(1, 1)$ or $(-1, -1)$. When θ is close to -1, trajectories will approach $(1, -1)$ or $(-1, 1)$. If the transitions between high and low values of θ are fast enough, then the system will exhibit toggle behaviour.

For example if $\theta = sin(\omega t)$, then correct operation depends on the value of ω. As shown in figure 1, when $\omega = 1$, trajectories visit the neighbourhoods of $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$, but in counter-clockwise order. Using `AUTO` [5] to generate a bifurcation diagram, we observed a bifurcation between $\omega = 0.241$ and $\omega = 0.242$. With $\omega = 0.241$, the "spiral" portion of each trajectory is long enough that trajectories visit the four neighbourhoods described above in clockwise order. There is another bifurcation between $\omega = 3.876$ and $\omega = 3.877$. With $\omega \geq 3.877$, there is a single fixed-point attractor at the origin.

III. Brockett's Annulus

In the previous section, inputs of the toggle were constant or sinusoidal. Of course, the signals occurring in a real chip are much more complicated and can only be approximated using such simple functions. To verify the operation of the toggle for realistic inputs we need a way to characterise entire classes of possible input functions.

We describe signals using Brockett's annulus construction [4] as shown in figure 5. For a signal, ϕ, the annulus gives a relation that must hold between ϕ and its time derivative, $\dot{\phi}$. When ϕ is in region 1 of the annulus, its value is constrained to lie between ϕ_{ll} and ϕ_{lh}. These give the minimum and maximum values for a logically low signal. Because the sign of $\dot{\phi}$ is unconstrained in region 1, ϕ can remain low arbitrarily long. When the signal leaves region 1, it must increase and enter region 2. In region 2, the signal monotonically increases until it reaches region 3. The minimum rise time corresponds to a trajectory that follows the outer boundary of the ring, and the maximum rise time corresponds to a trajectory along the inner boundary. Region 3 corresponds to a logically high signal, and region 4 describes a monotonically falling signal.

We add two non-geometric requirements to Brockett's constraints. First, Brockett's construction allows a signal to spend an arbitrarily small amount of time in the high or low regions of the ring. Signals that "ricochet" off the high or low region are unrepresentative of the signals that occur in real circuits. Accordingly, we add explicit constraints for minimum high and low times. Second, we require that any valid signal has no last transition. This eliminates some tedious, special cases in the analysis.

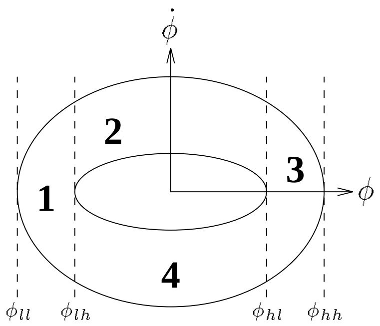

Fig. 5. Brockett's annulus

A Brockett annulus describes an infinite class of signals. In the next section, we present a particular annulus and show that our toggle operates correctly for any input satisfying the constraints of this annulus. We show that the output of the toggle satisfies the same annulus constraints as we require of the inputs. Thus, an output of one toggle can be used as an input to another. This justifies our construction for a binary counter.

IV. Verification

To verify the toggle, we present a Brockett annulus and show that for any input in this annulus, the toggle has an invariant set with twice the period of this input. We then show that the x and y outputs of the toggle satisfy the constraints of the same Brockett annulus.

The derivative functions we consider are infinitely differentiable; therefore, a trajectory is uniquely determined by its initial point [6]. A set of points is an invariant set if every trajectory that starts in the set remains in the set forever. By analogy with the period of an attractor in a non-autonomous system [7], we say that an invariant set, I, has period p with respect to some signal, ϕ, if there exists a constant c such that:

1. The intersection of I with the $\phi = c$ hyperplane consists of $2p$ distinct regions.
2. There exists an ordering of these regions, $M_0, \ldots, M_{2p-1}$ such that each trajectory in I visits these regions cyclically in this order.

The $\phi = c$ hyperplane separates the invariant set into $2p$ distinct subsets. These subsets correspond to the discrete states of a digital interpretation of the system.

To verify the toggle, we specify a Brockett annulus, B, and show that there exists a period-2 invariant set such that the outputs of the toggle satisfy B. Because the invariant set is period-2, the system functions as a toggle. Because the outputs of the toggle satisfy the constraints required of the input, the toggles may be connected in a chain to form a binary counter. This provides the framework for verification.

The toggle from section II does not satisfy these conditions, and a slight modification was necessary. The problem is that during any transition of θ, the output that is supposed to retain its logical value during the transition will dip slightly towards the origin. The dip is large enough to preclude using the same Brockett annulus for θ and x or y. The solution is to map the input function to one with shorter rise and fall times. In particular, we verified the system:

$$\begin{aligned} \theta &= \tanh(4\phi) \\ \dot{x} &= (1+\theta)(x - x^3) + (1-\theta)(-y - x) \\ \dot{y} &= (1+\theta)(x - y) + (1-\theta)(y - y^3) \end{aligned} \qquad (4)$$

Where ϕ is the input to the toggle and x and y are the outputs. The mapping $\theta = \tanh(4\phi)$ can be thought of as a simplistic model for a saturating amplifier with a maximum gain of four.

We verified the toggle modeled by equation 4 using a Brockett annulus whose boundaries are ellipses centered on the origin and with axes parallel to the ϕ and $\dot{\phi}$ axes. The inner annulus has a ϕ radius of 0.6, and a $\dot{\phi}$ radius of 0.5. The outer annulus has a ϕ radius of 1.1, and a $\dot{\phi}$ radius of 10.0. The minimum dwell times in the high and low regions are both 2 time units. We will call an input function **valid** if it satisfies these constraints.

The properties of the toggle can be verified starting with a bounding box for the initial region and computing bounding boxes as ϕ completes two cycles to determine the bounding box at the end of two cycles. If the final bounding box is contained in the initial box, then the union of the boxes forms an invariant set. As this approach overestimates the reachable space, it is conservative: it can fail to verify a correct system, but it will not falsely verify an incorrect system. Let $\phi_{\min}$, $\phi_{\max}$, $x_{\min}$, $x_{\max}$, $y_{\min}$, and $y_{\max}$ define a bounding box in the obvious way. We note that $\dot{x}$ is negative monotonic in y; for fixed x and y, $\dot{x}$ is linear in θ and therefore monotonic in ϕ. To compute $\dot{x}_{\min}$, the time derivative of the left edge of the box, we compute the minimum of $\dot{x}$ according to equation 4 at $(\phi_{\min}, x_{\min}, y_{\max})$ and $(\phi_{\max}, x_{\min}, y_{\max})$. The computations of $\dot{x}_{\max}$, $\dot{y}_{\min}$, and $\dot{y}_{\max}$ are similar. We obtain bounding boxes for the system of equation 4 by integrating the equations for the derivatives of the bounds of the bounding box using a fourth order Runga-Kutta integrator [8].

To verify the existence of an invariant set, we chose an initial region with ϕ in the logically low region of the Brockett annulus, $0.85 \le x \le 1.02$, and $-1.02 \le y \le -0.9$. Let Q_0 denote this set of points. Using the methods described above, it is straightforward to show that for Q_0, $\dot{x}_{\min} > 0$, $\dot{x}_{\max} < 0$, $\dot{y}_{\min} > 0$, and $\dot{y}_{\max} < 0$. Thus, for any valid input function, trajectories will remain in Q_0 as long as ϕ remains in the logically low region. When ϕ is in the rising region of the annulus, we exploit the monotonicity of ϕ and integrate the bounding box equations with respect to ϕ. For any valid input function, we have upper and lower bounds for $\dot{\phi}$ as a function of ϕ. These allow us to convert time derivatives of the bounds of the bounding box to derivatives with respect to ϕ. Integrating in this fashion, we obtain $x_{\min} > 0.86$, $x_{\max} < 1.02$, $y_{\min} > -0.80$, and $y_{\max} < 0.97$ when ϕ enters the logical high region. Any valid input function must remain in the logically high region for at least two time units. Integrating the bounds with respect to time for these two time units

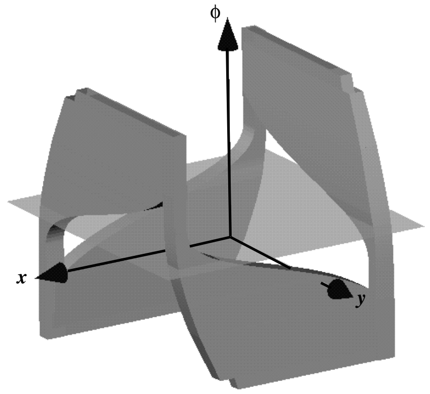

Fig. 6. An invariant set for the toggle

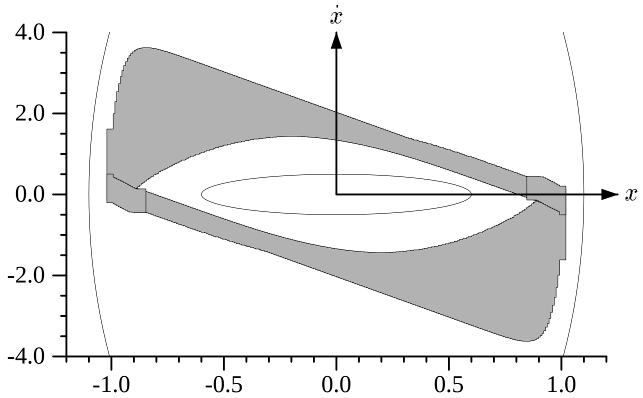

Fig. 7. Toggle output satisfies Brockett annulus

yields $x_{\min} > 0.99$, $x_{\max} < 1$, $y_{\min} > 0.96$, $y_{\max} < 1$. Let Q_0' denote the set of points with x and y in these intervals and ϕ in the logical high region. For all valid input functions, any trajectory starting in Q_0 leads to a point in Q_0' by the time that ϕ leaves the logical high region. Furthermore, all such trajectories are contained in the union of the bounding boxes determined by the integrations described above. Let I_0 be the set of points in this union of bounding boxes.

The analysis is completed by noting the symmetry of the toggle. The nextphase function rotates a set of points 90 degrees counter-clockwise and in negates the value of ϕ; the allphases computes the union of nextphase over four consecutive clock phases:

$$\begin{aligned} \mathsf{nextphase}(Q) &= \{(\phi, x, y) | (-\phi, -y, x) \in Q\} \\ \mathsf{allphases}(Q) &= \textstyle\bigcup_{k=0}^{\infty} \mathsf{nextphase}^k(Q) \end{aligned} \tag{5}$$

Note that $Q_0' \subset \mathsf{nextphase}(Q_0)$. Let $I = \mathsf{allphases}(I_0)$. By construction, trajectories starting in Q_0 remain in I forever. Thus, I is an invariant set. Also, note that for any set Q, $\mathsf{nextphase}^4(Q) = Q$ which yields

$$\mathsf{allphases}(Q) \;=\; \textstyle\bigcup_{k=0}^{3} \mathsf{nextphase}^k(Q) \tag{6}$$

The set I_0 is determined by numerical integration; after which, equation 6 provides a simple way to determine I. The rest of the verification is achieved by inspecting the set I. As shown in figure 6, I intersects the $\phi = 0$ plane in four distinct regions, and all trajectories cycle through those regions in the same order. Thus, I is period-2 with respect to ϕ. By computing bounds on $\dot{x}$ and $\dot{y}$ for the bounding box obtained at each step of the integration, we can verify that x and y also satisfy the Brockett constraints. If a bounding box is particularly long in one dimension, we first slice it into thinner slabs and check each slab separately. Figure 7 shows x that and $\dot{x}$ (and by symmetry, y and $\dot{y}$) satisfy the Brockett constraints: the ellipses of the original annulus are drawn, and the bounding region for $(x, \dot{x})$ is shaded. Finally, inspection of I shows that the minimum dwell times are at least 4.1 time units, thus, the outputs of the toggle satisfy all of the constraints required of the inputs.

V. Conclusions

We have presented a simple dynamical system that implements the discrete behaviour of a toggle flip-flop. The attractor structure of the model was explored. The model is infinitely differentiable; thus, the toggle demonstrates how a common digital behaviour can be obtained from a completely smooth continuous system. Signal specifications and the mappings from continuous to discrete interpretations were based on Brockett's annulus construction. Because the toggle's outputs satisfy the constraints for the inputs, toggle elements can be coupled in a chain to form a binary counter. Our analysis for the simple toggle applies to these counters with phase-spaces of arbitrarily high dimension.

VI. Acknowledgements

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