The Continual Reachability Set and its Computation Using Maximal Reachability Techniques

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Abstract— **The continual reachability set, the set of initial states of a constrained dynamical system that can reach a target at any desired time, is introduced. The properties of this set are investigated and its connection with maximal reachability constructs is examined. Owing to this connection, efficient and scalable maximal reachability techniques can be used to compute the continual reachability set. An approximation of this set based on ellipsoidal techniques is presented. The results are demonstrated on a problem of control of anesthesia.**

I. Introduction

In many cyberphysical systems, mathematical guarantees of safety or performance are critical to efficient and effective operation. Reachability analysis has typically been used to provide guarantees of safety (collision avoidance, flight envelope protection, etc.) for systems in which hard constraints must be observed despite bounded control authority (e.g. [1]–[5]). Reachability analysis identifies the states backward (forward) reachable by a constrained dynamical system from a given target (initial) set of states. However, such constraints may often be temporarily relaxed in favor of improved overall performance. For example, consider a fleet of environmental monitoring motes which must be dispersed to gather relevant data (e.g., water clarity, air quality), but power conservation is paramount; or a remotely operated fleet of ground, air, or sea vehicles, in which a desired formation must be achieved within a certain time frame.

In this paper, we focus on the application of backward reachability analysis to performance problems, as well as its relationship to safety problems. *Maximal* and *minimal* reachability constructs, formally introduced in [6], relate the ability of a system to reach a target and the types of controllers required. In formation of the maximal reachability construct, the input tries to steer as many states as possible to the target set. In formation of the minimal reachability construct, the trajectories reach the target set regardless of the input applied. The objects generated by each of these constructs have unique properties: The maximal reachability construct can be used to synthesize inputs that steer the trajectories to the target, while the minimal reachability construct can be used to synthesize inputs that keep the trajectories away from the target. Two other closely related constructs are the *invariance* and *viability kernels* [7] that describe the behavior of the trajectories within the target set itself depending on how the input is applied. All these fundamentally different constructs are computed using two separate categories of algorithms: *Lagrangian methods* (e.g. [8]–[11]) that follow the trajectories and *Eulerian methods* (e.g. [3], [12]–[14]) that are based on gridding the state space.

Here our main contribution is to highlight an overlooked reachability construct that we refer to as the *continual reachability set*, which can be used to provide guarantees of performance in reachability analysis. The continual reachability set is the set of states that can reach the target at any given time within the finite horizon. For any state in this set, there exists at least one input policy that can steer the trajectory emanating from that state to the target at any desired time. Initiating the system from this set provides additional flexibility to a supervisory controller to choose a policy that optimizes a trade-off between the desired time-to-reach the target and the input effort (or other performance indices) required to drive the state to the target.

We are motivated by a problem of guaranteed performance and safety in control of anesthesia. A variety of approaches to controlling depth of anesthesia have been proposed [15]–[20] to improve patient recovery, lessen anesthetic drug usage, and reduce time spent at drug saturation levels. We identify the set of states for which an anesthesiologist (or a closed-loop controller) can choose the most suitable drug infusion rate, while ensuring that the patient reaches the desired clinical effect at any given time. The continual reachability set provides a guarantee of performance, in that for any initial state in the set, the desired clinical effect can be reached at arbitrary times. This may be particularly useful in scenarios in which one wishes to minimize the total administered drug, or to achieve a desired depth of anesthesia arbitrarily fast.

Section II defines various backward reachability constructs including the continual reachability set. Section III establishes the connections between these constructs and expresses the continual reachability set in terms of maximal reachability sets. This allows us to use Lagrangian methods to compute the continual reachability set. We formulate an approximation of this set based on ellipsoidal techniques [21] and show the results on a problem of control of anesthesia in Section IV. Section V provides concluding remarks.

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II. Backward Constructs for Constrained Dynamical Systems

Consider a continuous time, continuously valued system

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0 \tag{1}$$

with state space $\mathcal{X} := \mathbb{R}^n$, state vector $x(t) \in \mathcal{X}$, and input $u(t) \in \mathcal{U}$ where $\mathcal{U}$ is a compact and convex subset of $\mathbb{R}^m$. We assume that the vector field $f \colon \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is Lipschitz in x and continuous in u. Denote by $\mathscr{U}_{[0,t]}$ the set of Lebesgue measurable functions $u(\cdot)$ from $[0,t]$ to $\mathcal{U}$. With an arbitrary time horizon $\tau > 0$, for every $t \in [0,\tau]$, $x_0 \in \mathcal{X}$, and $u(\cdot) \in \mathscr{U}_{[0,t]}$, there exists a unique trajectory $\xi_{x_0,0,u(\cdot)} \colon [0,t] \to \mathcal{X}$ that satisfies the initial condition $\xi_{x_0,0,u(\cdot)}(0) = x_0$ and the differential equation (1) almost everywhere. (Here the subscript 0 denotes the initial time.)

For a nonempty state constraint (target) set $\mathcal{K} \subseteq \mathcal{X}$ the following *backward* constructs can be defined:

Definition 1 (Maximal Reachability Set): The maximal reachability set at time t is the set of initial states for which there exists an input $u(\cdot)$ such that the trajectories emanating from those states reach $\mathcal{K}$ exactly at time t:

$$Reach^{\sharp}_{t}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 2 (Maximal Reachability Tube): The maximal reachability tube (also known as the *possible victory domain* [13], *attainability tube* [8], or *capture basin* [22]) over the horizon $[0,\tau]$ is the set of initial states for which there exists an input such that the trajectories emanating from those states reach $\mathcal{K}$ at some time $t \in [0,\tau]$:

$$Reach^{\sharp}_{[0,\tau]}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \exists t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 3 (Minimal Reachability Set): The minimal reachability set at time t is the set of initial states such that, for every input $u(\cdot)$, the trajectories emanating from those states reach $\mathcal{K}$ exactly at time t:

$$Reach^{\flat}_{t}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 4 (Minimal Reachability Tube): The minimal reachability tube (also known as the *certain victory domain* [13]) over the horizon $[0,\tau]$ is the set of initial states such that, for every input $u(\cdot)$, the trajectories emanating from those states reach $\mathcal{K}$ at some time $t \in [0,\tau]$:

$$Reach^{\flat}_{[0,\tau]}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \exists t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 5 (Invariance Kernel): The (finite horizon) invariance kernel of $\mathcal{K}$ is the set of initial states in $\mathcal{K}$ such that the trajectories emanating from those states remain within $\mathcal{K}$ for all time $t \in [0,\tau]$ for all input $u(\cdot)$:

$$Inv_{[0,\tau]}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \forall u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \forall t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 6 (Viability Kernel): The (finite horizon) viability kernel (also known as the *largest controlled-invariant* subset) of $\mathcal{K}$ is the set of all initial states in $\mathcal{K}$ for which there exists a $u(\cdot)$ such that the trajectories emanating from those states remain within $\mathcal{K}$ for all time $t \in [0,\tau]$:

$$Viab_{[0,\tau]}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \exists u(\cdot) \in \mathscr{U}_{[0,\tau]},\, \forall t \in [0,\tau],\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

Definition 7 (Continual Reachability Set): The continual reachability set defined over the time horizon $[0,\tau]$ is the set of initial states in $\mathcal{K}$ for which, for any given time $t \in [0,\tau]$, there exists a $u(\cdot)$ such that the trajectories emanating from those states reach $\mathcal{K}$ at t:

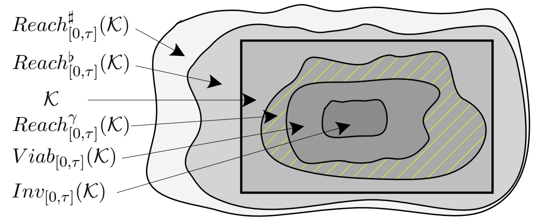

Fig. 1. An illustration of the inclusions in Proposition 1.

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) := \{x_0 \in \mathcal{X} \mid \forall t \in [0,\tau],\, \exists u(\cdot) \in \mathscr{U}_{[0,t]},\, \xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}\}.$$

What differentiates these constructs from one another is the *type* and *order* of quantifiers that operate on the time and input variables. These subtle differences generate fundamentally distinct sets. The states that belong to each of these sets have properties that are unique to that set. The properties of the invariance and viability kernels as well as the maximal and minimal reachability sets and tubes have been studied extensively [3], [7], [23]. To examine the properties of the continual reachability set let us first establish the connections between all of the above constructs.

III. Connections Between Backward Constructs

We begin by stating a generic inclusion relation (Fig. 1):

Proposition 1:

$$Inv_{[0,\tau]}(\mathcal{K}) \subseteq Viab_{[0,\tau]}(\mathcal{K}) \subseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \subseteq \mathcal{K} \subseteq Reach^{\flat}_{[0,\tau]}(\mathcal{K}) \subseteq Reach^{\sharp}_{[0,\tau]}(\mathcal{K}). \tag{2}$$

Proof: That $Inv_{[0,\tau]}(\mathcal{K}) \subseteq Viab_{[0,\tau]}(\mathcal{K}) \subseteq \mathcal{K}$ is well-known [7]. To show $Viab_{[0,\tau]}(\mathcal{K}) \subseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$, take $x_0 \in Viab_{[0,\tau]}(\mathcal{K})$. Therefore, $(\exists u(\cdot) \in \mathscr{U}_{[0,\tau]})(\forall t \in [0,\tau])\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \implies (\forall t \in [0,\tau])(\exists u(\cdot) \in \mathscr{U}_{[0,t]})\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \iff x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$. To show $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \subseteq \mathcal{K}$, take $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ and let $\tau = 0$. x_0 must also belong to $\mathcal{K}$. To show $\mathcal{K} \subseteq Reach^{\flat}_{[0,\tau]}(\mathcal{K})$ take $x_0 \in \mathcal{K}$ and let $\tau = 0$. Thus $\xi_{x_0,0,u(\cdot)}(0) = x_0$ for any $u(\cdot) \in \mathscr{U}_{[0,\tau]}$. Therefore, $x_0 \in Reach^{\flat}_{[0,\tau]}(\mathcal{K})$. To prove $Reach^{\flat}_{[0,\tau]}(\mathcal{K}) \subseteq Reach^{\sharp}_{[0,\tau]}(\mathcal{K})$, take $x_0 \in Reach^{\flat}_{[0,\tau]}(\mathcal{K})$. Therefore, $(\forall u(\cdot) \in \mathscr{U}_{[0,\tau]})(\exists t \in [0,\tau])\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \implies (\exists u(\cdot) \in \mathscr{U}_{[0,\tau]})(\exists t \in [0,\tau])\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \iff x_0 \in Reach^{\sharp}_{[0,\tau]}(\mathcal{K})$. ■

The following proposition describes the connection between reachability tubes and viability and invariance kernels.

Proposition 2 ([13], [1]): With $\mathcal{K}^c$ denoting the complement of $\mathcal{K}$ in $\mathcal{X}$ we have

$$Reach^{\sharp}_{[0,\tau]}(\mathcal{K}^c) = (Inv_{[0,\tau]}(\mathcal{K}))^c, \tag{3}$$

$$Reach^{\flat}_{[0,\tau]}(\mathcal{K}^c) = (Viab_{[0,\tau]}(\mathcal{K}))^c. \tag{4}$$

In addition, it has been shown in [6] (and indirectly in [1] using the Hamilton-Jacobi-Bellman framework) that the following connections exist between maximal and minimal reachability sets and tubes.

Proposition 3 ([6], [1]):

$$Reach^{\sharp}_{[0,\tau]}(\mathcal{K}) = \bigcup_{t\in[0,\tau]} Reach^{\sharp}_{t}(\mathcal{K}), \tag{5}$$

$$Reach^{\flat}_{[0,\tau]}(\mathcal{K}) \supseteq \bigcup_{t\in[0,\tau]} Reach^{\flat}_{t}(\mathcal{K}). \tag{6}$$

In fact, (5) is precisely how the Lagrangian methods compute the maximal reachability tube. That is, to compute $Reach^{\sharp}_{[0,\tau]}(\mathcal{K})$ these algorithms compute $Reach^{\sharp}_{t}(\mathcal{K})$ at every time step and then take their collective union.

Among Lagrangian methods, the technique in [24] is thus far the only method that has been extended to handle universally quantified inputs. Therefore, it is also capable of computing minimal reachability sets. As a by-product of this feature, the same technique can also be used to *directly* compute the invariance kernel.

Proposition 4 ([1]):

$$Inv_{[0,\tau]}(\mathcal{K}) = \bigcap_{t\in[0,\tau]} Reach^{\flat}_{t}(\mathcal{K}). \tag{7}$$

Proof: $x_0 \in \bigcap_{t\in[0,\tau]} Reach^{\flat}_{t}(\mathcal{K}) \iff (\forall t \in [0,\tau])(\forall u(\cdot) \in \mathscr{U}_{[0,\tau]})\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \iff (\forall u(\cdot) \in \mathscr{U}_{[0,\tau]})(\forall t \in [0,\tau])\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \iff x_0 \in Inv_{[0,\tau]}(\mathcal{K})$. This can also be verified from (3) and (5) and the simple fact that $Reach^{\flat}_{t}(\mathcal{K}) = (Reach^{\sharp}_{t}(\mathcal{K}^c))^c$. ■

Note however that due to (6), minimal reachability tubes (and viability kernels via Proposition 2) cannot be formed from minimal reachability sets. It is shown in [7] and [6] that $Viab_{[0,\tau]}(\cdot)$ and $Reach^{\flat}_{[0,\tau]}(\cdot)$ are the only constructs that can be used to prove the existence of an input (*control* in this context) which guarantees "safety" of the system over the horizon $[0,\tau]$. At this time, these constructs are only available from computationally costly Eulerian methods that rely on gridding the state space.

Theorem 1:

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) = \bigcap_{t\in[0,\tau]} Reach^{\sharp}_{t}(\mathcal{K}). \tag{8}$$

Proof: $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \iff (\forall t \in [0,\tau])(\exists u(\cdot) \in \mathscr{U}_{[0,t]})\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K} \iff (\forall t \in [0,\tau])x_0 \in Reach^{\sharp}_{t}(\mathcal{K}) \iff x_0 \in \bigcap_{t\in[0,\tau]} Reach^{\sharp}_{t}(\mathcal{K})$. ■

Remark 1: Due to Theorem 1 the (scalable and efficient) Lagrangian techniques can be used to compute the continual reachability sets.

A. Properties of the Continual Reachability Set

- P1. If (1) is a linear system and $\mathcal{K}$ is convex and compact, then $Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ is also convex and compact. This is due to the fact that for a linear system with convex and compact target, the maximal reachability sets are all convex and compact [23]. Therefore, their intersection is also convex and compact.
- P2. $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \subseteq Reach^{\sharp}_{t}(\mathcal{K})$, $\forall t \in [0,\tau]$. Particularly, if $Reach^{\sharp}_{t}(\mathcal{K}) = \emptyset$ for some t then $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) = \emptyset$.
- P3. The states that lie outside of the continual reachability set but inside the maximal reachability tube can only reach the target at specific times (provided that an appropriate input is applied to the system); e.g., a state $x_0 \in Reach^{\sharp}_{[0,\tau]}(\mathcal{K}) \setminus \bigcup_{t\in[0,\tau]\setminus\{\tilde{t}\}} Reach^{\sharp}_{t}(\mathcal{K})$ can only reach the target at time $\tilde{t}$; or, $x_0 \in \big(Reach^{\sharp}_{\hat{t}}(\mathcal{K}) \cap Reach^{\sharp}_{\tilde{t}}(\mathcal{K})\big) \setminus \bigcup_{t\in[0,\tau]\setminus\{\hat{t},\tilde{t}\}} Reach^{\sharp}_{t}(\mathcal{K})$ can only reach $\mathcal{K}$ at times $\hat{t}$ and $\tilde{t}$. In contrast, $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ can reach the target at any given desired time. (Fig. 2)

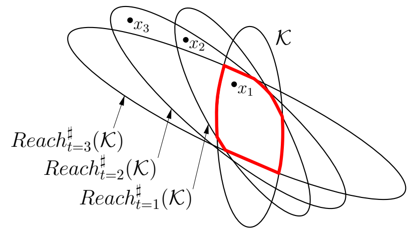

Fig. 2. A time-discretized illustration of P3. The maximal reachability sets at times $t = 1, 2, 3$ are shown. The state x_1 which belongs to the continual reachability set (outlined in red) can reach the target at any desired time, whereas x_2 can be steered to the target at times $t = 1$ and $t = 2$, and x_3 can only reach the target at $t = 2$.

- P4. The states in $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \setminus Viab_{[0,\tau]}(\mathcal{K})$ *may* temporarily leave $\mathcal{K}$, but will eventually return to it at the desired time (provided that an appropriate input is applied). It is clear from Theorem 1 that the farthest achievable distance for a trajectory that emanates from one such state is determined by the maximal reachability set it belongs to. Let $d(x,\mathcal{A}) := \inf_{a\in\mathcal{A}}\|x-a\|$ denote the distance between a point $x \in \mathcal{X}$ and a compact subset $\mathcal{A}$ of $\mathcal{X}$. Also let $d_{H_1}(\mathcal{S},\mathcal{A}) := \sup_{s\in\mathcal{S}} \inf_{a\in\mathcal{A}}\|s-a\|$ denote the one-sided Hausdorff distance from a compact subset $\mathcal{S}$ to $\mathcal{A}$ in $\mathcal{X}$. Then for every $x_0 \in Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ and some $t \in [0,\tau]$, for all $\hat{t} \in [0,t]$,

$$d(\xi_{x_0,0,u(\cdot)}(\hat{t}),\mathcal{K}) \le d_{H_1}\big(Reach^{\sharp}_{t-\hat{t}}(\mathcal{K}),\mathcal{K}\big) \tag{9}$$

for any $u(\cdot) \in \mathscr{U}_{[0,t]}$ such that $\xi_{x_0,0,u(\cdot)}(t) \in \mathcal{K}$.

Notice that P3 implies that when the constraint set $\mathcal{K}$ represents a target performance of the system and the initial state is in the continual reachability set, the controller can decide the most suitable course of action by applying an appropriate control law based on the desired time to reach the target and the required control effort. For example, depending on the circumstances, the controller may choose a more aggressive control action in favor of reaching the target at a shorter time span. This provides the controller with additional degrees of freedom in choosing the appropriate closed-loop trajectory while ensuring that the target is reached.

B. Dealing With Under- and Over-Approximations

Most Lagrangian techniques compute maximal reachability sets by under- and/or over-approximations. Since the states that lie outside of the continual reachability set may not reach $\mathcal{K}$ at a desired time, an under-approximation of this set is the correct form of approximation. With under-approximations of the maximal reachability sets the continual reachability set can be correctly under-approximated, ensuring that all states in the approximating set possess the properties of the continual reachability set.

Corollary 1: Let $\mathcal{A}_{\downarrow}$ denote an under-approximation of $\mathcal{A}$.

$$Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq \bigcap_{t\in[0,\tau]} Reach^{\sharp}_{t}(\mathcal{K}_{\downarrow})_{\downarrow}. \tag{10}$$

C. Approximating the Continual Reachability Set Using Ellipsoidal Techniques

Assume that (1) has linear dynamics and can therefore be restated as a linear differential inclusion $\dot{x}(t) \in Ax(t) + B\mathcal{U}$ for a.e. $t \in [0, \tau]$, with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Assume further that $\mathcal{U}$ is a nonempty compact ellipsoid in $\mathbb{R}^m$.

We will show in this section that when $\mathcal{K}$ is (or can be reasonably under-approximated by) a nonempty compact ellipsoid, ellipsoidal techniques [21] can be used to compute an approximation of $Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$.

Definition 8 ([23]): An ellipsoid with center $q \in \mathbb{R}^n$ and shape matrix $Q \in \mathbb{R}^{n\times n}$, $Q = Q^T \succ 0$, is defined as

$$\mathcal{E}(q, Q) := \left\{ x \in \mathbb{R}^n \mid \left\langle (x - q), Q^{-1}(x - q)\right\rangle \leq 1 \right\}. \quad (11)$$

Let $\mathcal{K}_{\downarrow\varepsilon} := \mathcal{E}(x_\tau, X_\tau) \subset \mathcal{X}$ be an ellipsoidal under-approximation of $\mathcal{K}$, and $\mathcal{U} = \mathcal{E}(p, P) \subset \mathbb{R}^m$. Then, as in [24] the maximal reachability set at time t is a compact convex set whose center evolves according to $\dot{x}^*(t) = Ax^*(t) + Bp$, $x^*(\tau) = x_\tau$. For a given *direction* $\ell_\tau \in \mathbb{R}^n$ consider the solution $\ell(t)$ to the adjoint equation $\dot{\ell}(t) = -A^T\ell(t)$, $\ell(\tau) = \ell_\tau$. There exists a "tight" [8] *internal approximating* ellipsoid $\mathcal{E}(x^*(t), X^-_\ell(t))$ in the direction of $\ell(t)$ such that

$$Reach^{\sharp}_t(\mathcal{K}_{\downarrow\varepsilon}) \supseteq \mathcal{E}(x^*(t), X^-_\ell(t)) \quad (12)$$

with a shape matrix $X^-_\ell(t)$ obtained from a differential equation [24]. Since this under-approximation is tight for every $\ell(t)$, it is shown that with $\mathcal{L} := \{l \in \mathbb{R}^n \mid \|l\| = 1\}$,

$$Reach^{\sharp}_t(\mathcal{K}_{\downarrow\varepsilon}) = \bigcup_{\ell_\tau\in\mathcal{L}} \mathcal{E}(x^*(t), X^-_\ell(t)). \quad (13)$$

Proposition 5:

$$\begin{aligned} Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) &\supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \\ &= \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau\in\mathcal{L}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right). \end{aligned} \quad (14)$$

Proof: The equality can be verified by substituting (13) in Theorem 1, noting that the approximation of $Reach^{\sharp}_t(\mathcal{K}_{\downarrow\varepsilon})$ is exact for a.e. $t \in [0, \tau]$. The inclusion stems from the fact that for any two sets $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \subseteq \mathcal{B}$, we have $Reach^{\gamma}_{[0,\tau]}(\mathcal{A}) \subseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{B})$. ■

In practice, only a finite number of directions can be used for maximal reachability set computations. Let $\mathcal{V}$ be a subset of $\mathcal{L}$ with finite cardinality. Therefore, for every $t \in [0, \tau]$,

$$Reach^{\sharp}_t(\mathcal{K}_{\downarrow\varepsilon}) \supseteq Reach^{\sharp}_t(\mathcal{K}_{\downarrow\varepsilon})_{\downarrow} := \bigcup_{\ell_\tau\in\mathcal{V}} \mathcal{E}(x^*(t), X^-_\ell(t)). \quad (15)$$

Corollary 2:

$$\begin{aligned} Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) &\supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \\ &\supseteq \bigcap_{t\in[0,\tau]} \left(\bigcup_{\ell_\tau\in\mathcal{V}} \mathcal{E}(x^*(t), X^-_\ell(t)) \right). \end{aligned} \quad (16)$$

Furthermore, with $x^*_t := x^*(t)$ and $X^-_{\ell,t} := X^-_\ell(t)$,

$$\begin{aligned} &Reach^{\gamma}_{[0,\tau]}(\mathcal{K}) \supseteq Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon}) \supseteq \\ &\left\{ x \in \mathcal{X} \,\middle|\, \sup_{t\in[0,\tau]} \min_{\ell_\tau\in\mathcal{V}} \left\langle (x - x^*_t),\, (X^-_{\ell,t})^{-1}(x - x^*_t)\right\rangle \leq 1 \right\}. \end{aligned} \quad (17)$$

Proof: Substituting (15) in (10) yields (16). The expression (17) is a direct consequence of (16). ■

Remark 2: From a numerical standpoint, it is not necessary to compute intersections and unions of ellipsoids to obtain an under-approximation of the continual reachability set; it is easy to determine whether a given point in the state space belongs to $Reach^{\gamma}_{[0,\tau]}(\mathcal{K})$ by evaluating (17).

D. Continual Reachability Set for Discrete-Time Systems

The results presented in the previous sections, including Corollary 2, hold in their entirety for discrete-time systems as well. The only exception is how an internal approximating ellipsoid is computed; cf. [25] for more details.

IV. Example: Control of Anesthesia

Over the past few years, an interdisciplinary team of researchers at the University of British Columbia has been developing a closed-loop drug delivery system for anesthesia. A bolus-based Neuromuscular Blockade Advisory System has been developed and experimentally validated (in open-loop) [26], [27]. This system is based on data from more than 80 patients via clinical trials. LTI models of dynamical response to the bolus have been numerically identified for each patient by exploiting a Laguerre model framework [28]. A key element for fully closing the loop, and obtaining regulatory certification, are guarantees of safety and performance that the viability kernel and the continual reachability set can provide. The results presented here represent work towards that goal.

Consider the problem of computing the continual reachability set for a dynamical system that represents a patient under anesthesia subject to a therapeutic target.

A. Patient Model and Constraints

Consider the following discrete-time LTI system describing the Laguerre dynamics of a patient [26], [28]:

$$x(t + 1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \quad (18)$$

with time step $t \in \mathbb{Z}^+$, state vector $x(t) \in \mathbb{R}^6$, input (rocuronium infusion rate [mg/kg/min]) $u(t) \in \mathbb{R}$, and output (pseudo-occupancy, a metric related to the patient's plasma concentration of anesthetic e.g. rocuronium or propofol) $y(t) \in \mathbb{R}$. The sampling interval is 20s and the system matrices are given in (∗).

The target is specified in terms of the pseudo-occupancy level: $y(t) \in \mathcal{K}_0 := [0.1, 1]$. The input is bounded above and below by hard physical constraints: $u(t) \in \mathcal{U}_0 := [0, 0.8]$, $\forall t \in [0, T]$, where T is the time step horizon.

B. Reformulating the Problem

Notice that this problem differs from a typical reachability formulation in two ways: 1) the target is given in the output space as opposed to the state space, and 2) the output y should track a reference that lies within $\mathcal{K}_0$. We reformulate the problem by first projecting the output bounds onto the state space and then making the control action regulatory. For brevity, we drop the time argument from the state, input, and output notations.

$$A=\begin{bmatrix} 0.9960 & 0 & 0 & 0 & 0 & 0 \\ 0.0080 & 0.9960 & 0 & 0 & 0 & 0 \\ -0.0080 & 0.0080 & 0.9960 & 0 & 0 & 0 \\ 0.0079 & -0.0080 & 0.0080 & 0.9960 & 0 & 0 \\ -0.0079 & 0.0079 & -0.0080 & 0.0080 & 0.9960 & 0 \\ 0.0079 & -0.0079 & 0.0079 & -0.0080 & 0.0080 & 0.9960 \end{bmatrix}, \quad B=\begin{bmatrix} 0.0894 \\ -0.0890 \\ 0.0886 \\ -0.0883 \\ 0.0879 \\ -0.0876 \end{bmatrix}, \quad C^T=\begin{bmatrix} 18.5000 \\ 8.2300 \\ 3.5300 \\ 4.3400 \\ 3.7000 \\ 3.0700 \end{bmatrix} \qquad (*)$$

1) Projection of Bounds onto the State Space: Consider the (non-singular) linear transformation

$$\begin{bmatrix} C \\ \mathbf{0}_{5\times 1} \quad I_5 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & \cdots & x_6 \end{bmatrix}^T =: \begin{bmatrix} w_1 & w_2 & \cdots & w_6 \end{bmatrix}^T.$$

The states $w_2, \dots, w_6$ are the Laguerre states $x_2, \dots, x_6$. In the new coordinate space, the bounds are state space constraints on the first state $w_1 := Cx = y$.

2) Tracking vs. Regulating: We perform an affine change of coordinates and shift the equilibrium point to the origin. This is done by augmenting the state vector in the w-space with the reference output signal y^* and applying a basis translation so that in the new coordinates the first state $w_1 := y$ becomes $z_1 := y - y^*$:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ w_2 \\ \vdots \\ w_6 \\ y^* \end{bmatrix} = \begin{bmatrix} y - y^* \\ w_2 \\ \vdots \\ w_6 \\ y^* \end{bmatrix} =: \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_6 \\ z_7 \end{bmatrix}. \qquad (19)$$

Let $u(t) = u_{ss}$ be the steady state control input needed for tracking a constant setpoint $y^*(t) = y^* = 0.9$. This value can be easily calculated using a standard state-feedback procedure from

$$\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix} = \begin{bmatrix} A - I & B \\ C & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ y^* \end{bmatrix}, \qquad (20)$$

where x_{ss} denotes the steady state equilibrium. To complete the reformulation, we deduct u_{ss} from the control set of the transformed system.

The new constraints for the transformed, extended system $z(t+1) = \widetilde{A}z(t) + \widetilde{B}u(t)$, $y(t) = \widetilde{C}z(t)$, with $\widetilde{A} \in \mathbb{R}^{7\times 7}$, $\widetilde{B} \in \mathbb{R}^{7\times 1}$, $\widetilde{C} \in \mathbb{R}^{1\times 7}$, are as follows:

$$\begin{aligned} z(t) &\in \mathcal{K} := (\mathcal{K}_0 - y^*) \times \mathbb{R}^6, \\ u(t) &\in \mathcal{U} := \mathcal{U}_0 - u_{ss}, \ \forall t \in [0, T]. \end{aligned} \qquad (21)$$

Note that with this formulation, the last state z_7 is allowed to take on values that are not needed; of actual interest are the behavior of the remaining states when $z_7 = y^*$.

C. Computing the Continual Reachability Set

We use the Ellipsoidal Toolbox [9] to under-approximate the continual reachability set of $\mathcal{K}$ based on the results presented in this paper. Notice that $\mathcal{U}$ is an ellipsoid. To under-approximate $\mathcal{K}$ with a non-degenerate ellipsoid $\mathcal{K}_{\downarrow\varepsilon}$ we use *a priori* knowledge about the typical values of the (Laguerre) states $z_2, \dots, z_6$ and bound them by an ellipsoid with spectral radius of $\lambda_{\max} = 2$ in those directions. (This imposed constraint can be relaxed if necessary.) We first

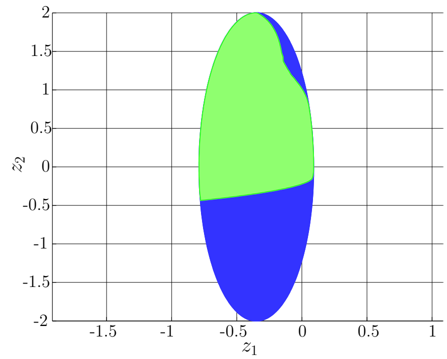

Fig. 3. The (z_1, z_2) projection of $\mathcal{K}_{\downarrow\varepsilon}$ (blue/dark) and $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$ (green/light) computed for patient #80 undergoing a 60 min surgery.

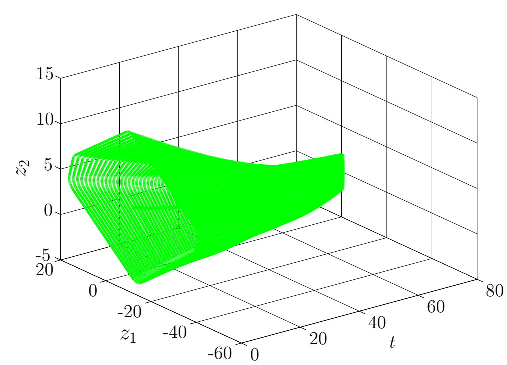

Fig. 4. The (z_1, z_2) projection of the backward maximal reachability tube $Reach^{\sharp}_{[0,T]}(\mathcal{K}_{\downarrow\varepsilon})_{\downarrow} = \bigcup_{t\in[0,T]} Reach^{\sharp}_{t}(\mathcal{K}_{\downarrow\varepsilon})_{\downarrow}$ computed for 80 time steps (each of 20s). Intersection of the time slices of this tube under-approximates $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$.

compute the maximal reachability sets of the system for $\mathcal{K}_{\downarrow\varepsilon}$ at every time step in 14 random directions for $T = 180$. We then under-approximate $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$ using Corollary 2.

Fig. 3 shows a projection of the constraint set $\mathcal{K}_{\downarrow\varepsilon}$ and its continual reachability set $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$ for the patient undergoing a 60 min surgery (180 time steps each of length 20s). The apparent non-convexity of $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$ is due to the fact that a limited number of directions has been used. The set approaches convexity when the number of directions goes to infinity, conforming to P1 and Proposition 5. Fig. 4 shows a projection of the maximal reachability sets for the first 80 time steps (in backward time) that are used when under-approximating $Reach^{\gamma}_{[0,\tau]}(\mathcal{K}_{\downarrow\varepsilon})$.

While to ensure safety it is usually desirable to keep the patient within the target clinical effect as much as possible, the required optimal drug infusion rate (or the resulting trajectory generated by such control policy) may not be physiologically ideal. There are instances in which, depending on the current physiological status of the patient, the anesthesiologist (or the closed-loop controller) may choose to relax the state constraint and allow the patient to temporarily leave the target in exchange for additional flexibility in selecting a better-suited (e.g. less aggressive, mildly varying) infusion rate or satisfying other secondary clinical objectives, while ensuring that the patient returns to the target at a prescribed time. The computed continual reachability set can be used for this purpose, as an alternative constraint for a model predictive controller. This will ensure that the closed-loop system chooses an infusion rate that is physiologically more optimized to meet the operating conditions and the patient's ability to handle the anesthetic drug, while simultaneously ensuring that the clinical effect reaches the target at a desired time. Such flexible, patient-oriented design has the benefit of tailoring the performance of the system towards the patient's needs during the surgery.

V. Conclusions

We studied the continual reachability set and its connection to other backward reachability constructs. Initiating the system from the continual reachability set provides the (supervisory) controller with an additional degree of freedom in choosing the most appropriate course of action. This is due to the fact that for every state in the continual reachability set and for every desired time-to-reach the target, there exists at least one input that drives the trajectory to the target at that desired time. Therefore, depending on circumstances, the controller can choose between reaching the target at a desired time or applying a more suitable input that also ensures that the target is reached. An under-approximation based on ellipsoidal techniques was formulated and applied to a problem of control of anesthesia. Future work includes i) synthesizing continual reachability control laws, and ii) accounting for model uncertainty in the computations.

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