Reach Sets and the Hamilton-Jacobi Equation

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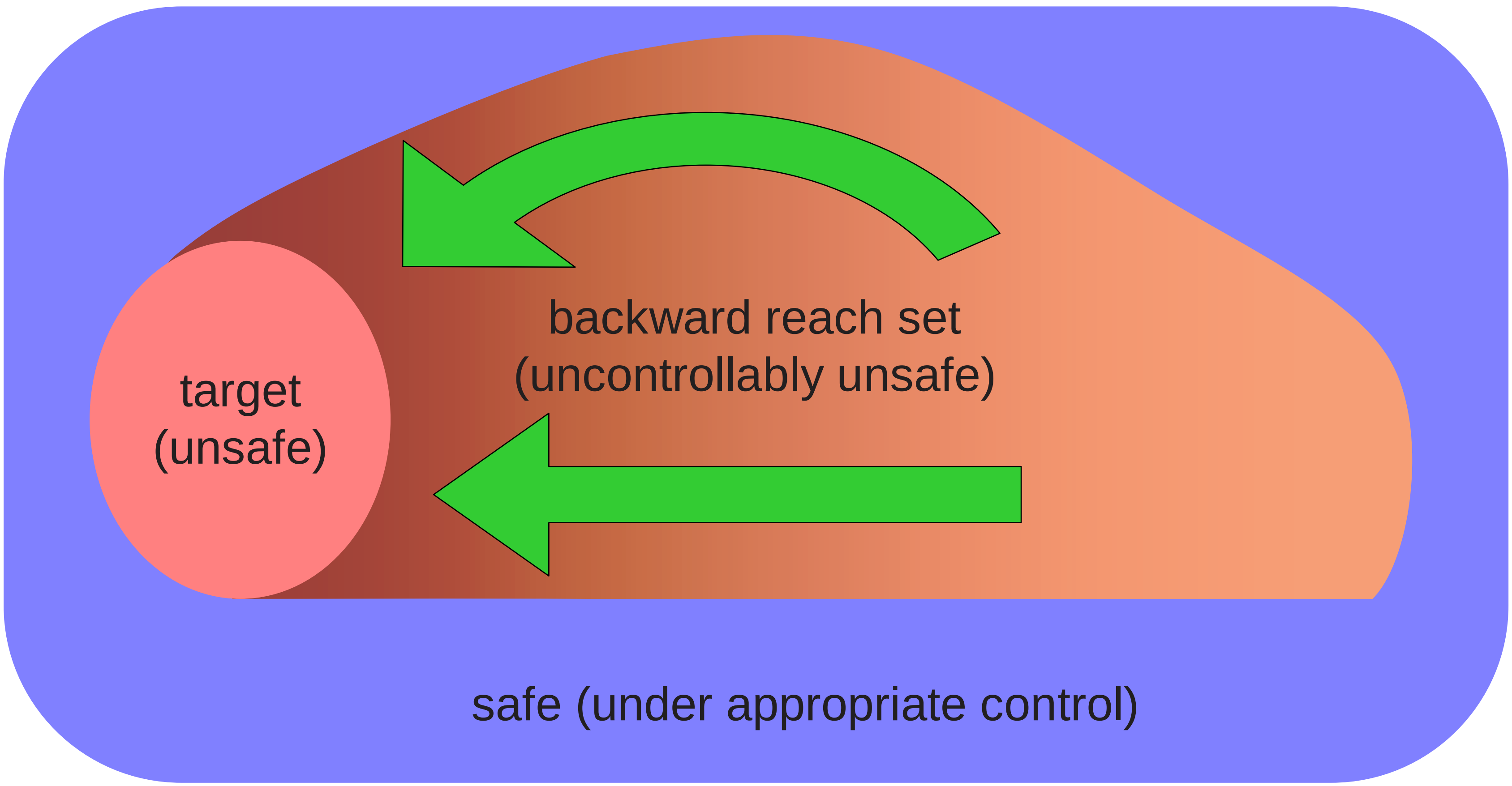
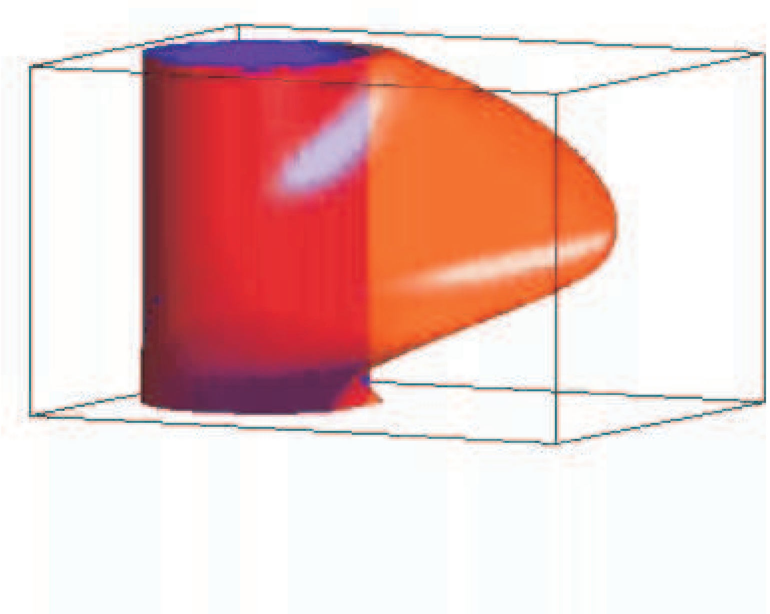
Reachable Sets: What and Why?

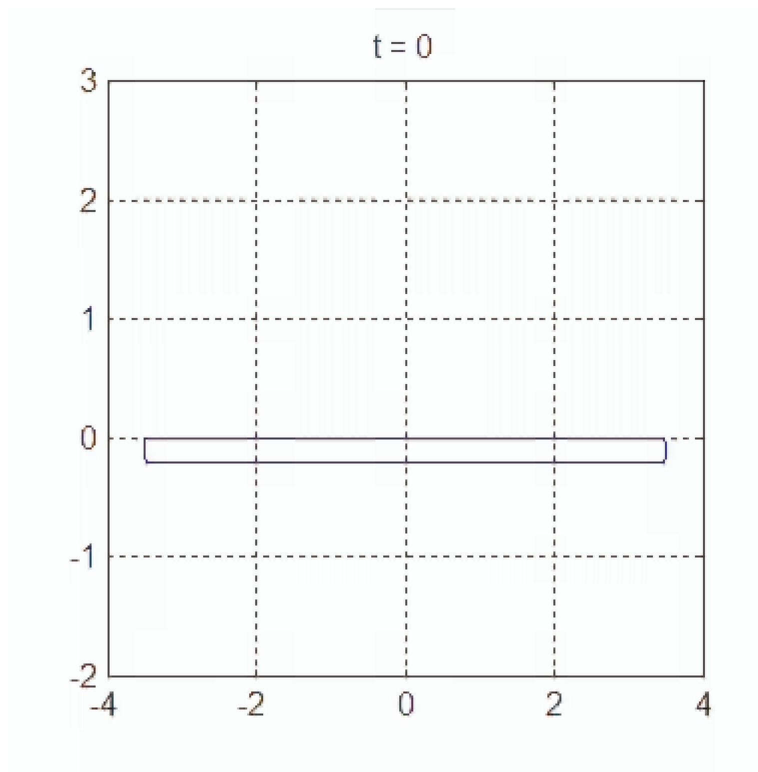
- One application: safety analysis
  - What states are doomed to become unsafe?
  - What states are safe given an appropriate control strategy?



Calculating Reach Sets

- Two primary challenges
  - How to represent set of reachable states
  - How to evolve set according to dynamics
- Discrete systems $x_{k+1} = \delta(x_k)$
  - Enumerate trajectories and states
  - Efficient representations: Binary Decision Diagrams
- Continuous systems $dx/dt = f(x)$?





Approaches to Continuous Reach Sets

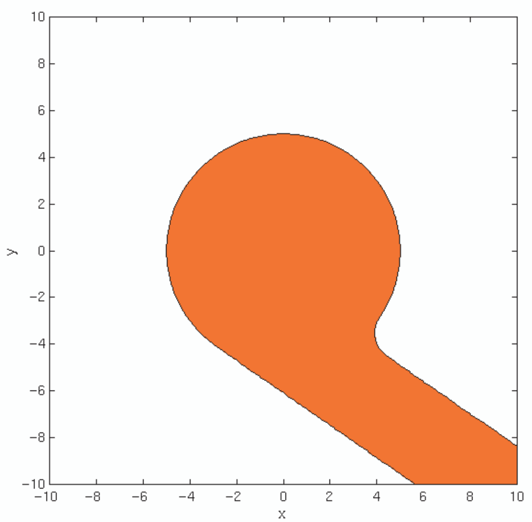
- Lagrangian approaches
  - Forward reach sets
  - Restricted class of dynamics
  - Restricted class of sets with compact representation
  - Guarantees of overapproximation
  - Examples: HyTech (Henzinger), Checkmate (Krogh), d/dt (Dang), ellipsoidal (Kurzhanski)
- Eulerian approaches
  - Backward reach sets
  - General dynamics including competitive inputs
  - General set shapes represented implicitly

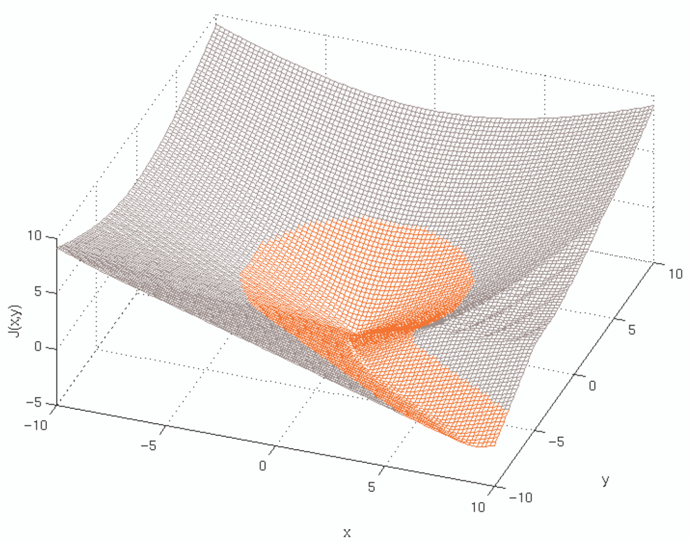
Implicit Surface Functions

- Set $G(t)$ is defined implicitly by an isosurface of a scalar function $\phi(x,t)$, with several benefits
  - State space dimension does not matter conceptually
  - Surfaces automatically merge and/or separate
  - Geometric quantities are easy to calculate

$$\phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$$

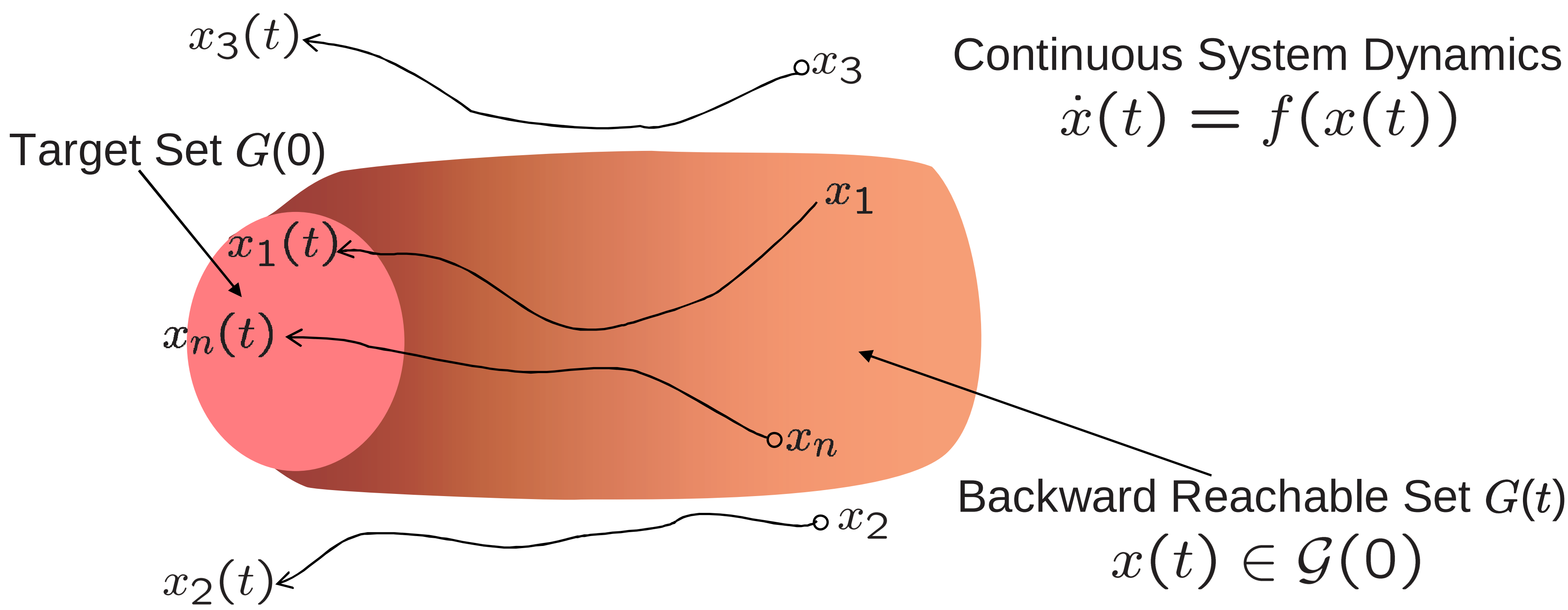
$$\mathcal{G}(t) = \{x \in \mathbb{R}^n \mid \phi(x,t) \leq 0\}$$





Continuous Backward Reachable Sets

- Set of all states from which trajectories can reach some given target state
  - For example, what states can reach $G(t)$?



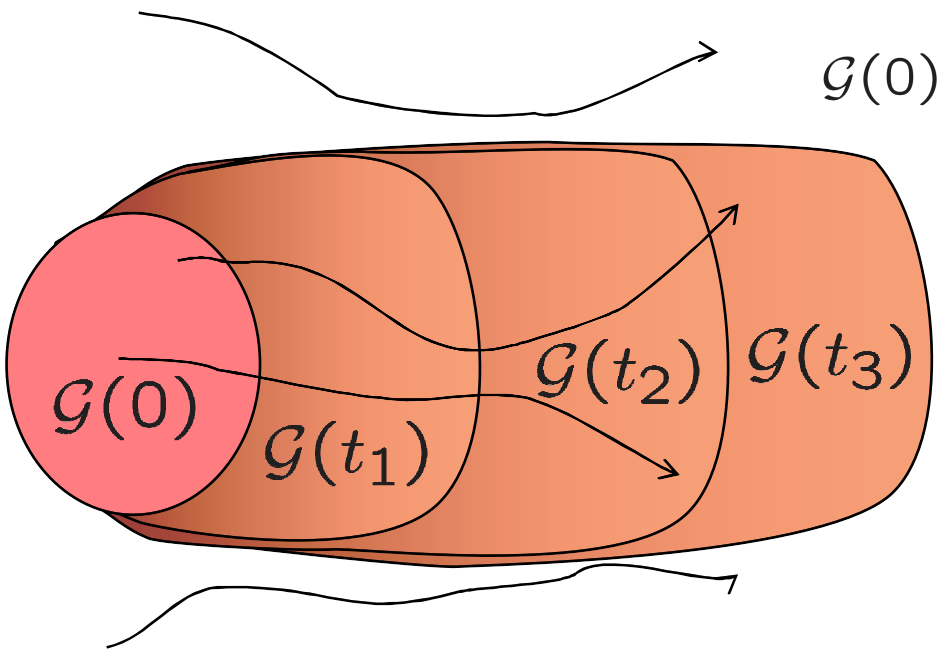
Why "Backward" Reachable Sets?

- To distinguish from forward reachable set
- To compute, run dynamics backwards in time from target set

$$\dot{x}(t) = -f(x(t))$$

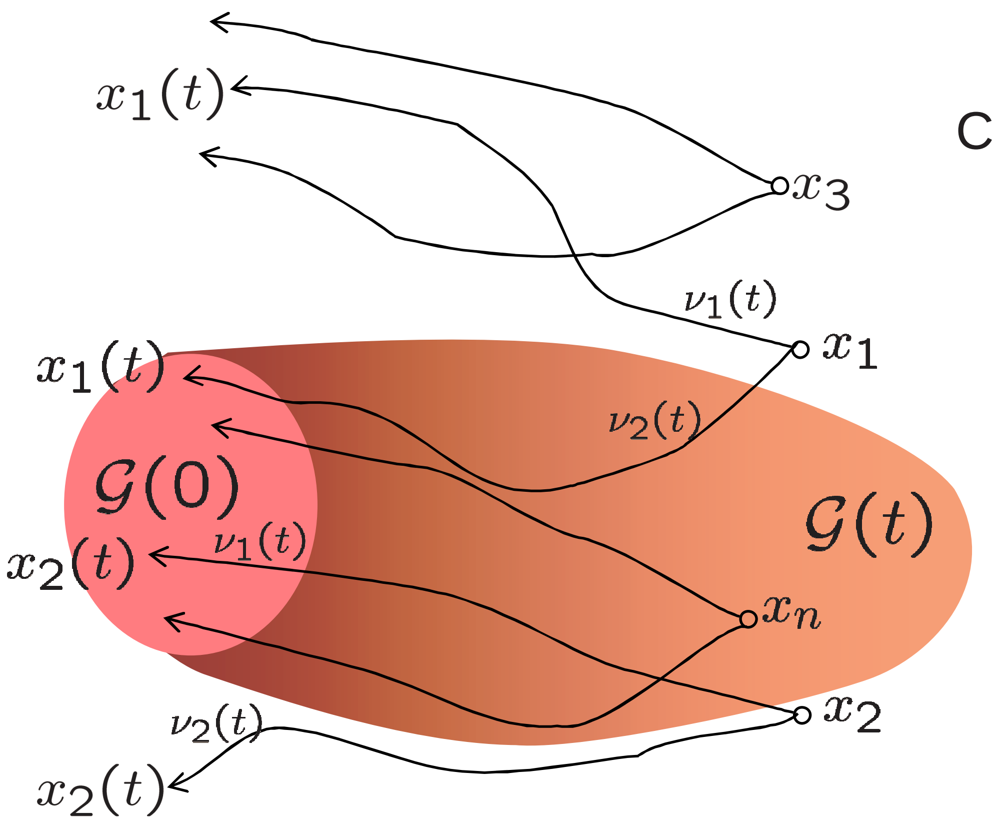
$$0 < t_1 < t_2 < t_3$$

$$\mathcal{G}(0) \subseteq \mathcal{G}(t_1) \subseteq \mathcal{G}(t_2) \subseteq \mathcal{G}(t_3)$$



Reachable Sets (controlled input)

- For most of our examples, target set is unsafe
- If we can control the input, choose it to avoid the target set
- Backward reachable set is unsafe no matter what we do



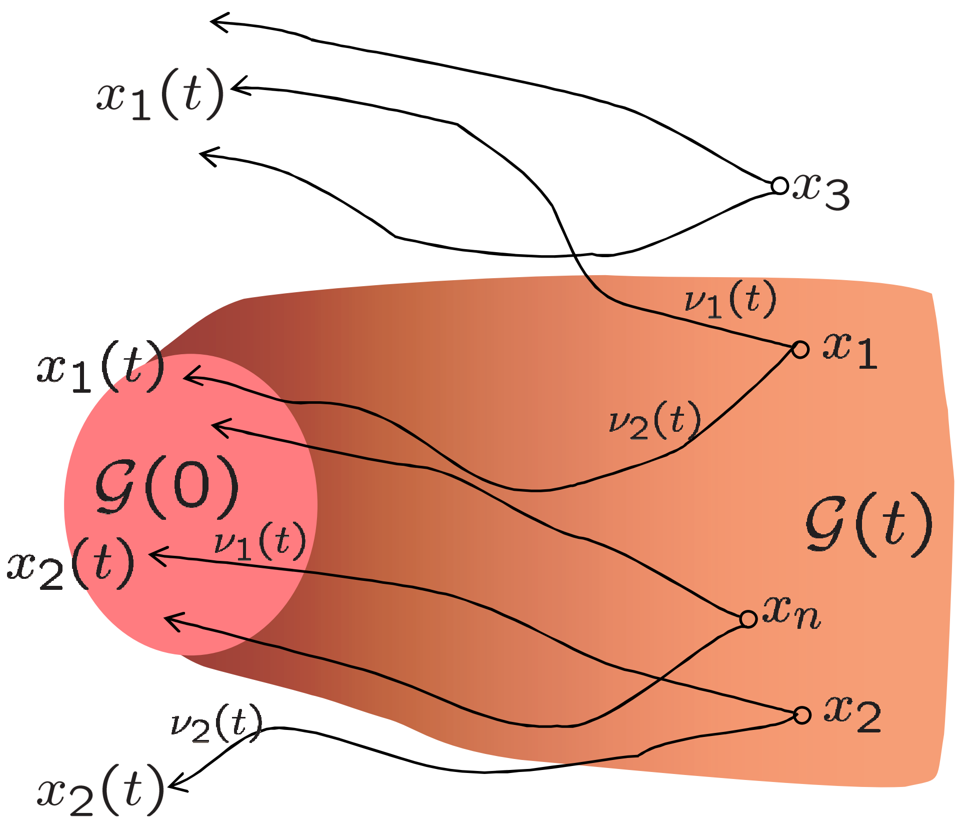
Continuous System Dynamics

$$\dot{x}(t) = f(x(t), \nu(t))$$

$$\forall \nu(\cdot), x(t) \in \mathcal{G}(0)$$

Reachable Sets (uncontrolled input)

- Sometimes we have no control over input signal
  - noise, actions of other agents, unknown system parameters
- It is safest to assume the worst case



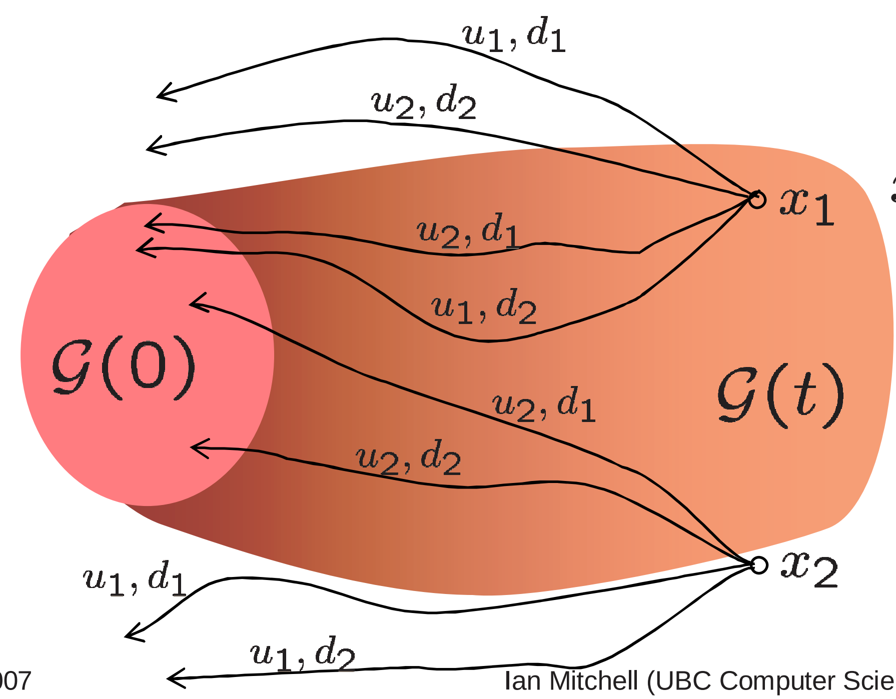
Continuous System Dynamics

$$\dot{x}(t) = f(x(t), \nu(t))$$

$$\exists \nu(\cdot), x(t) \in \mathcal{G}(0)$$

Two Competing Inputs

- For some systems there are two classes of inputs ν = (u,d)
  - Controllable inputs $u \in U$
  - Uncontrollable (disturbance) inputs $d \in D$
- Equivalent to a zero sum differential game formulation
  - If there is an advantage to input ordering, give it to disturbances



Continuous System Dynamics

$$\dot{x}(t) = f(x(t), u(t), d(t))$$

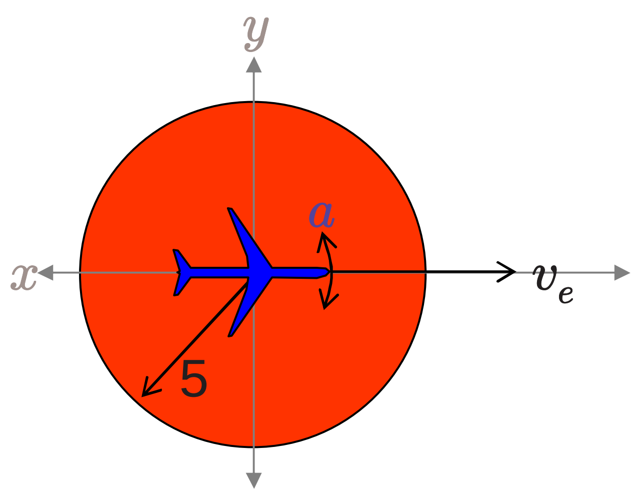
$$\forall u(\cdot), \exists d(\cdot), x(t) \in \mathcal{G}(0)$$

Game of Two Identical Vehicles

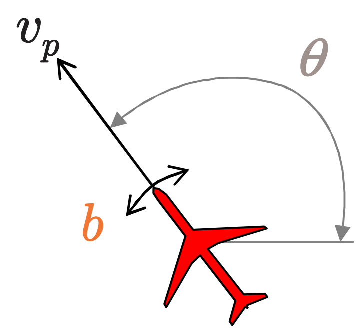
- Classical collision avoidance example
  - Collision occurs if vehicles get within five units of one another
  - Evader chooses turn rate $|a| \leq 1$ to avoid collision
  - Pursuer chooses turn rate $|b| \leq 1$ to cause collision
  - Fixed equal velocity $v_e = v_p = 5$

dynamics (pursuer)

$$\frac{d}{dt}\begin{bmatrix} x_p \\ y_p \\ \theta_p \end{bmatrix} = \begin{bmatrix} v_p \cos\theta_p \\ v_p \sin\theta_p \\ b \end{bmatrix}$$



evader aircraft (control)



pursuer aircraft (disturbance)

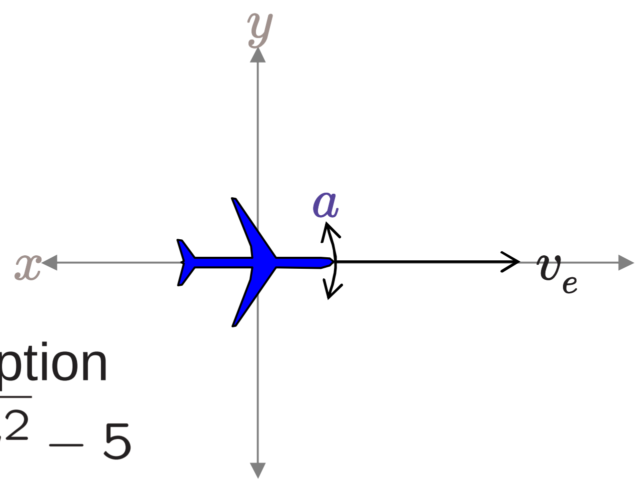
Collision Avoidance Computation

- Work in relative coordinates with evader fixed at origin
  - State variables are now relative planar location (x,y) and relative heading ψ

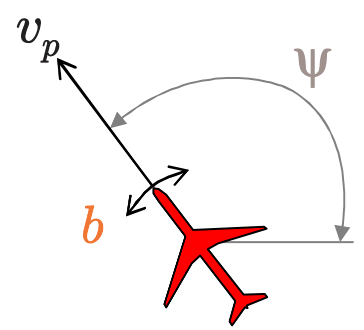
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} -v_e + v_p \cos\psi - ay \\ v_p \sin\psi - ax \\ b - a \end{bmatrix}$$

target set description

$$h(x) = \sqrt{x^2 + y^2} - 5$$



evader aircraft (control)



pursuer aircraft (disturbance)

Evolving Reachable Sets

- Modified Hamilton-Jacobi partial differential equation

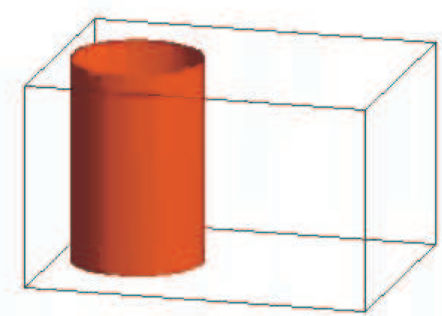
$$D_t\phi(x,t) + \min\left[0, H(x, D_x\phi(x,t))\right] = 0$$

$$\text{with Hamiltonian}: \; H(x,p) = \max_{a\in\mathcal{A}} \min_{b\in\mathcal{B}} f(x,a,b)\cdot p$$

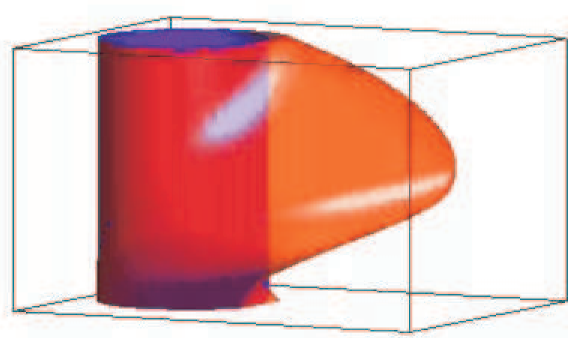
$$\text{and terminal conditions}: \; \phi(x,0) = h(x)$$

$$\text{where} \quad G(0) = \{x \in \mathbb{R}^n \mid h(x) \le 0\}$$

$$\text{and} \quad \dot{x} = f(x,a,b)$$



growth of reachable set



final reachable set

Time-Dependent Hamilton-Jacobi Eq'n

$$D_t\phi(x,t) + H(x, D_x\phi(x,t)) = 0$$

- First order hyperbolic PDE
  - Solution can form kinks (discontinuous derivatives)
  - For the backwards reachable set, find the "viscosity" solution [Crandall, Evans, Lions, …]
- Level set methods
  - Convergent numerical algorithms to compute the viscosity solution [Osher, Sethian, …]
  - Non-oscillatory, high accuracy spatial derivative approximation
  - Stable, consistent numerical Hamiltonian
  - Variation diminishing, high order, explicit time integration

Solving a Differential Game

- Terminal cost differential game for trajectories $\xi_f(\cdot;\, x, t, a(\cdot), b(\cdot))$

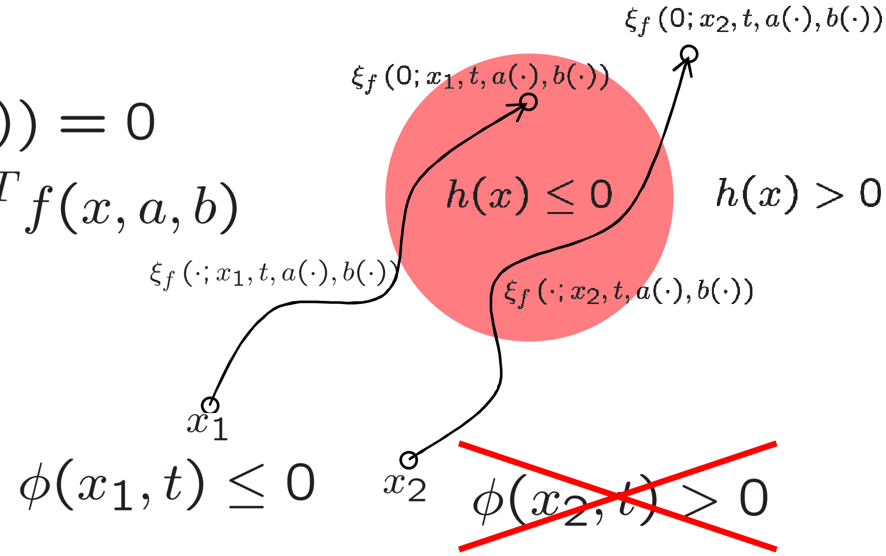
$$\phi(x,t) = \sup_{a(\cdot)} \inf_{b(\cdot)} h\left[\xi_f\left(0; x, t, a(\cdot), b(\cdot)\right)\right]$$

$$\text{where} \begin{cases} \xi_f\left(t; x, t, a(\cdot), b(\cdot)\right) = x \\ \dot{\xi}_f((s; x, t, a(\cdot), b(\cdot)) = f\left(x, a(s), b(s)\right) \\ \text{terminal payoff function } h(x) \end{cases}$$

- Value function solution $\phi(x,t)$ given by viscosity solution to basic Hamilton-Jacobi equation
  - [Evans & Souganidis, 1984]

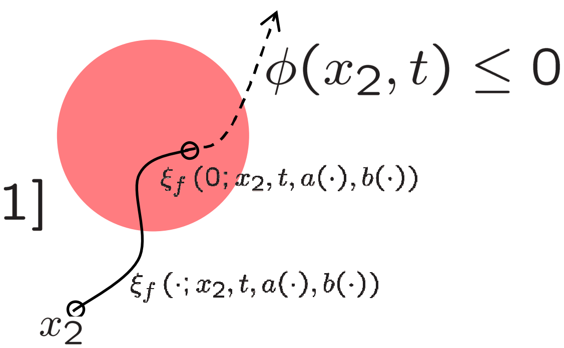
$$D_t\phi(x,t) + H\left(x, D_x\phi(x,t)\right) = 0$$

$$\text{where} \begin{cases} H(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x,a,b) \\ \phi(x,0) = h(x) \end{cases}$$



Modification for Optimal Stopping Time

- How to keep trajectories from passing through $G(0)$?
  - [Mitchell, Bayen & Tomlin 2004]
  - Augment disturbance input



$$\tilde{b} = \begin{bmatrix} b & \underline{b} \end{bmatrix} \text{ where } \underline{b} : [t, 0] \to [0, 1]$$

$$\tilde{f}(x, a, \tilde{b}) = \underline{b} f(x, a, b)$$

  - Augmented Hamilton-Jacobi equation solves for reachable set

$$D_t\phi(x,t) + \tilde{H}\left(x, D_x\phi(x,t)\right) = 0 \text{ where } \begin{cases} \tilde{H}(x,p) = \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T \tilde{f}(x, a, \tilde{b}) \\ \phi(x, 0) = h(x) \end{cases}$$

  - Augmented Hamiltonian is equivalent to modified Hamiltonian

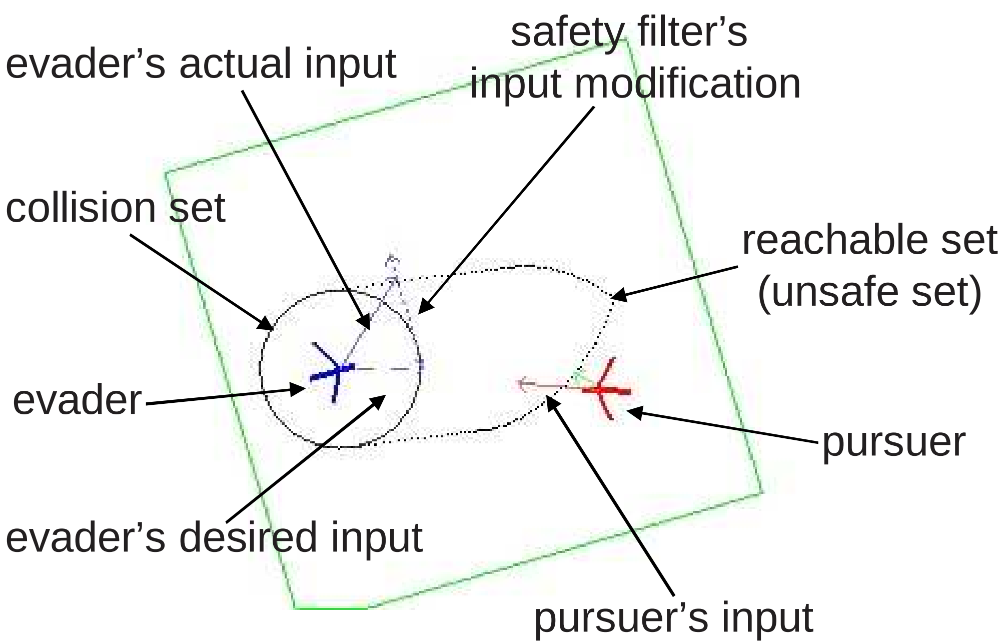
$$\begin{aligned} \tilde{H}(x,p) &= \max_{a \in \mathcal{A}} \min_{\tilde{b} \in \tilde{\mathcal{B}}} p^T \tilde{f}(x, a, \tilde{b}) \\ &= \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} \min_{\underline{b} \in [0,1]} \underline{b} p^T f(x, a, b) \\ &= \min \left[0, \max_{a \in \mathcal{A}} \min_{b \in \mathcal{B}} p^T f(x, a, b) \right] = \min\left[0, H(x,p)\right] \end{aligned}$$

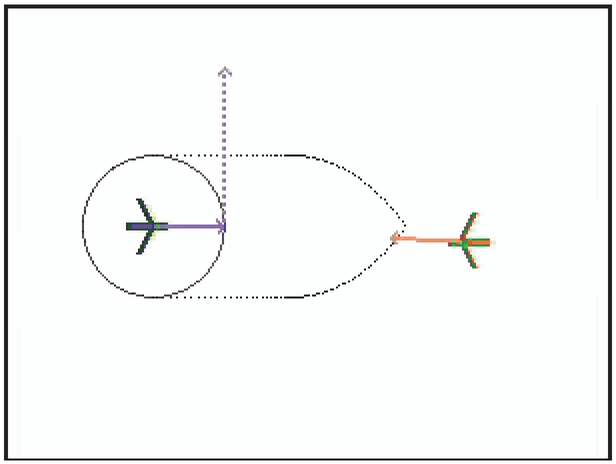
Alternative Eulerian Approaches

- Static Hamilton-Jacobi (Falcone, Sethian, …)
  - Minimum time to reach
  - (Dis)continuous implicit representation
  - Solution provides information on optimal input choices
- Viability kernels (Aubin, Saint-Pierre, …)
  - Based on set valued analysis for very general dynamics
  - Discrete implicit representation
  - Overapproximation guarantee
- Time-dependent Hamilton-Jacobi (this method)
  - Continuous solution
  - Information on optimal input choices available throughout entire state space
  - High order accurate approximations
- All three are theoretically equivalent

Application: Softwalls for Aircraft Safety

- Use reachable sets to guarantee safety
- Basic Rules
  - Pursuer: turn to head toward evader
  - Evader: turn to head east
- Evader's input is filtered to guarantee that pursuer does not enter the reachable set

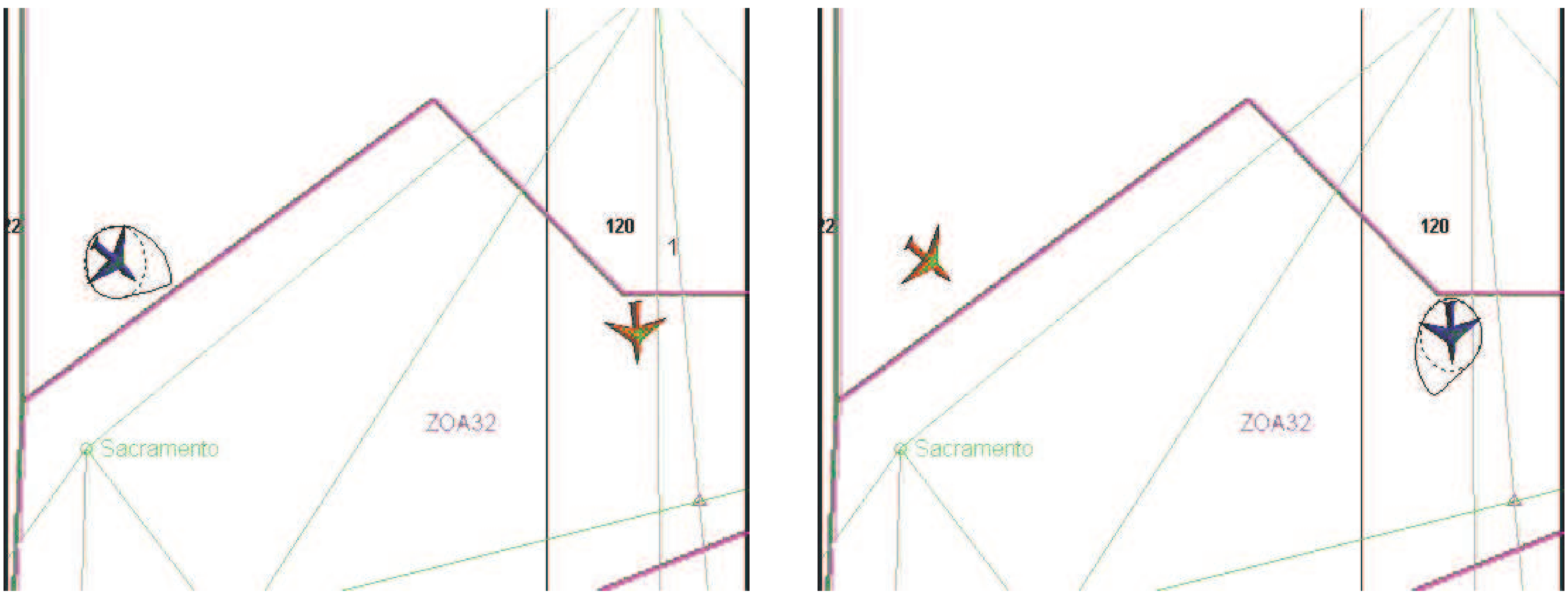




joint work with Edward Lee & Adam Cataldo

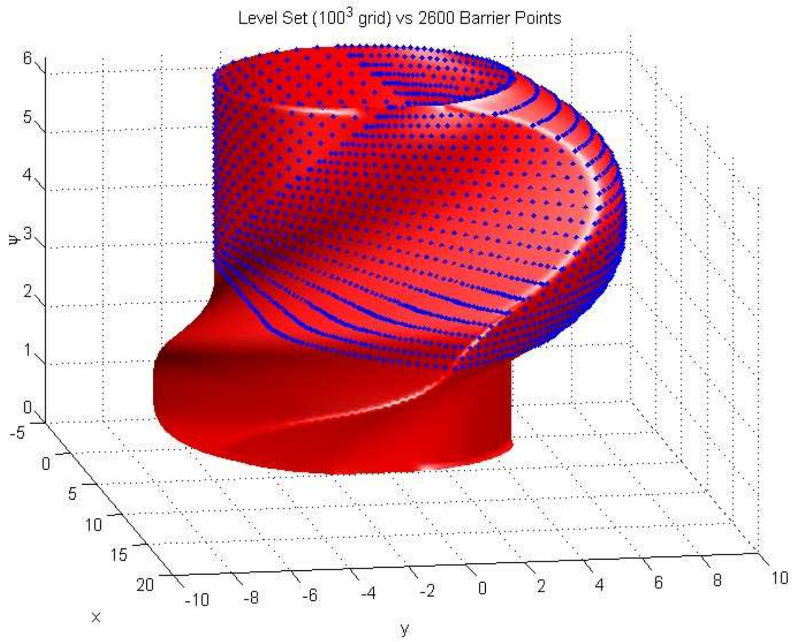
Application: Collision Alert for ATC

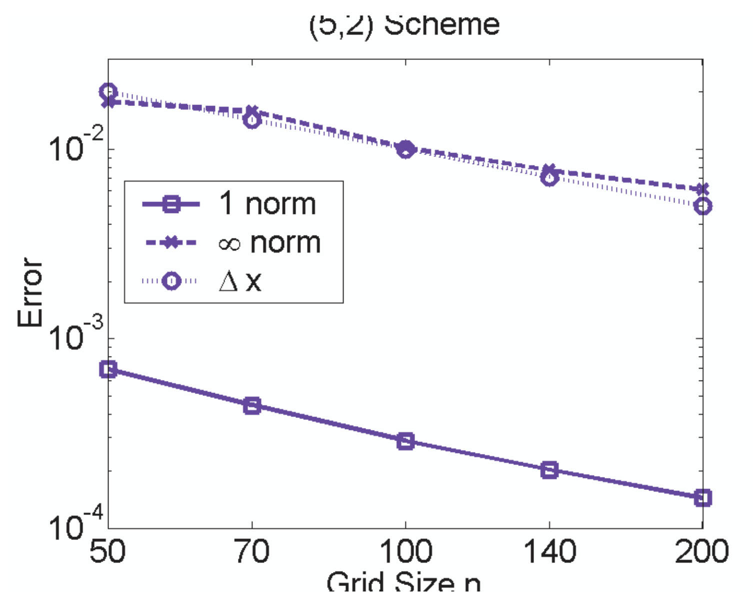
- Use reachable set to detect potential collisions and warn Air Traffic Control (ATC)
  - Find aircraft pairs in ETMS database whose flight plans intersect
  - Check whether either aircraft is in the other's collision region
  - If so, examine ETMS data to see if aircraft path is deviated
  - One hour sample in Oakland center's airspace—
    - 1590 pairs, 1555 no conflict, 25 detected conflicts, 2 false alerts



Validating the Numerical Algorithm

- Analytic solution for reachable set can be found [Merz, 1972]
  - Applies only to identical pursuer and evader dynamics
  - Merz's solution placed pursuer at the origin, game is not symmetric
  - Analytic solution can be used to validate numerical solution
  - [Mitchell, 2001]





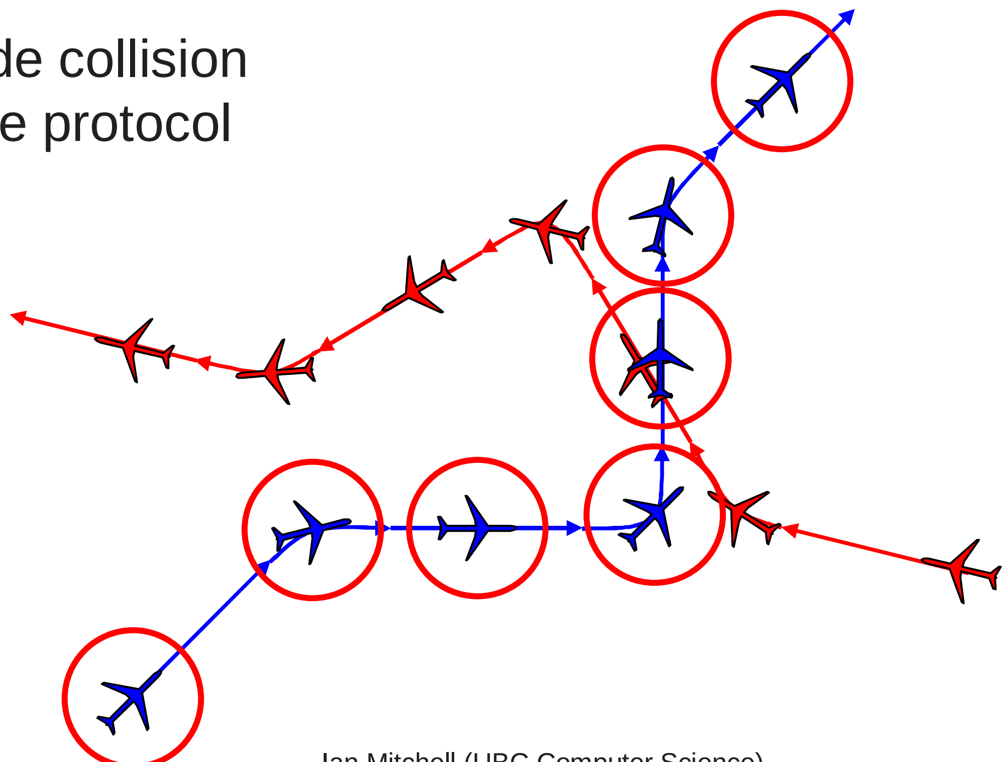
Hybrid System Reach Sets

Combining Continuous and Discrete Evolution

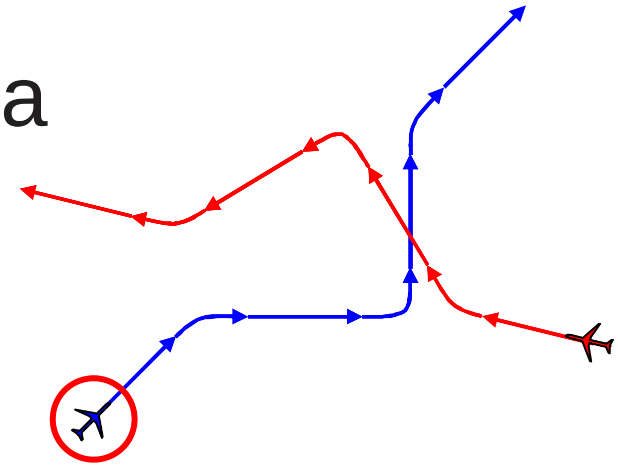
Why Hybrid Systems?

- Computers are increasingly interacting with external world
  - Flexibility of such combinations yields huge design space
  - Design methods and tools targeted (mostly) at either continuous or discrete systems
- Example: aircraft flight control systems

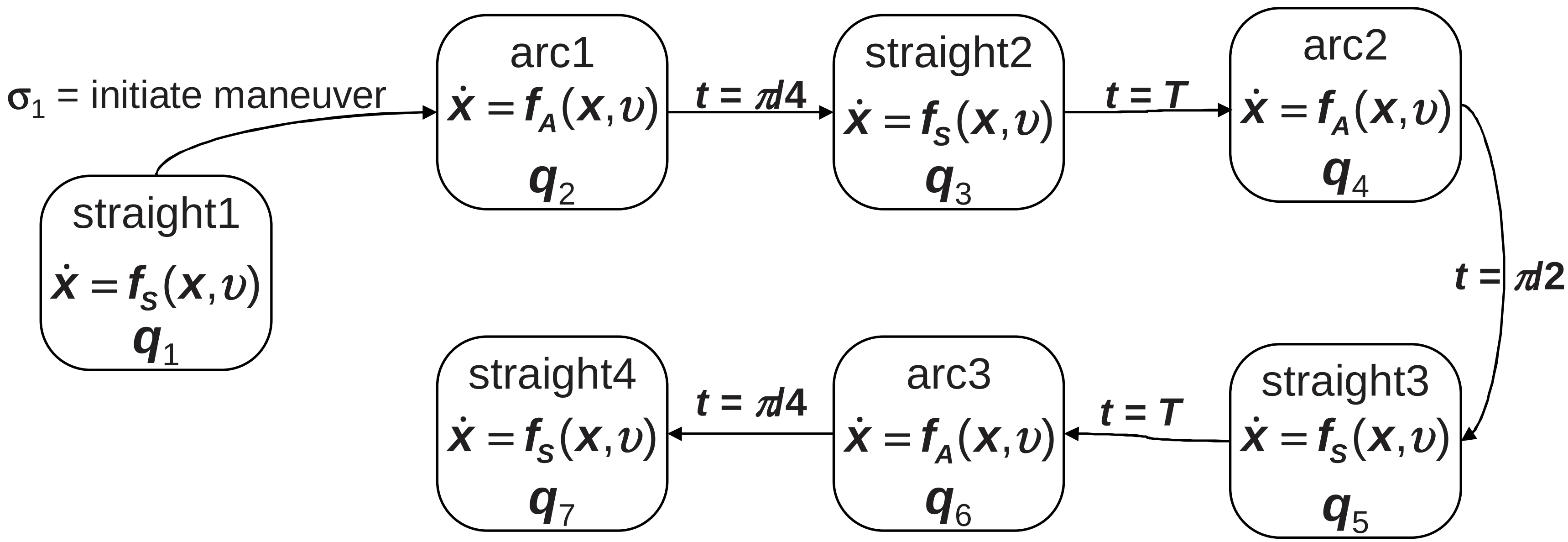
seven mode collision avoidance protocol



Hybrid Automata



- Discrete modes and transitions
- Continuous evolution within each mode



$$f_S\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -v + v\cos\psi \\ v\sin\psi \end{pmatrix}$$

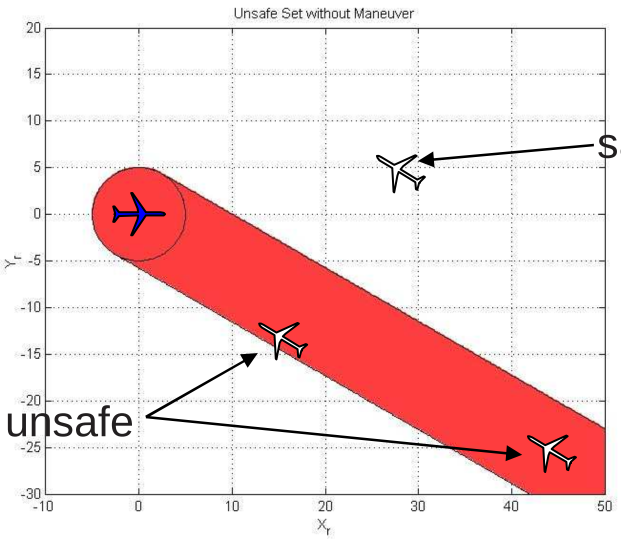
dynamics in straight modes

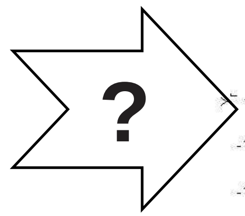
$$f_A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -v + v\cos\psi - x_2 \\ v\sin\psi + x_1 \end{pmatrix}$$

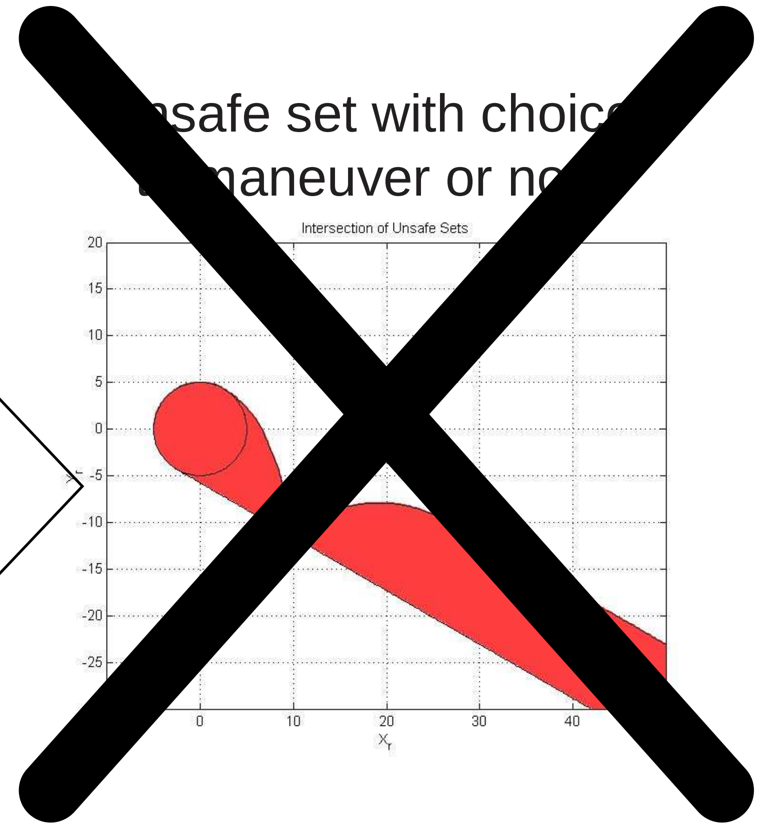
dynamics in arc modes

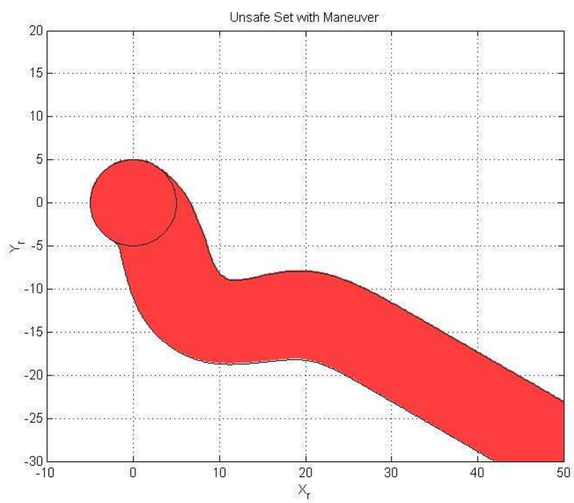
Seven Mode Safety Analysis

unsafe set without maneuver





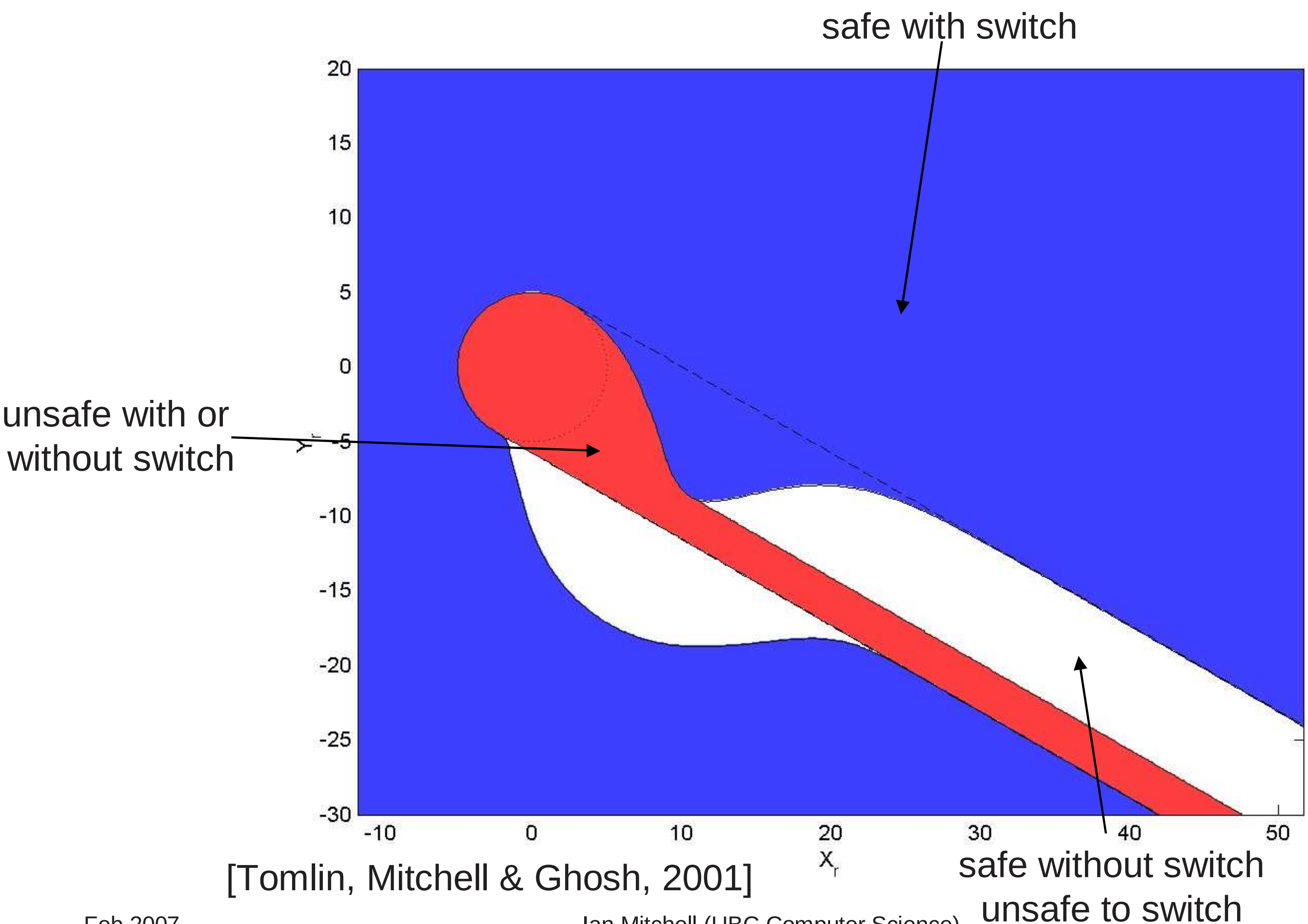




unsafe set with maneuver

Seven Mode Safety Analysis

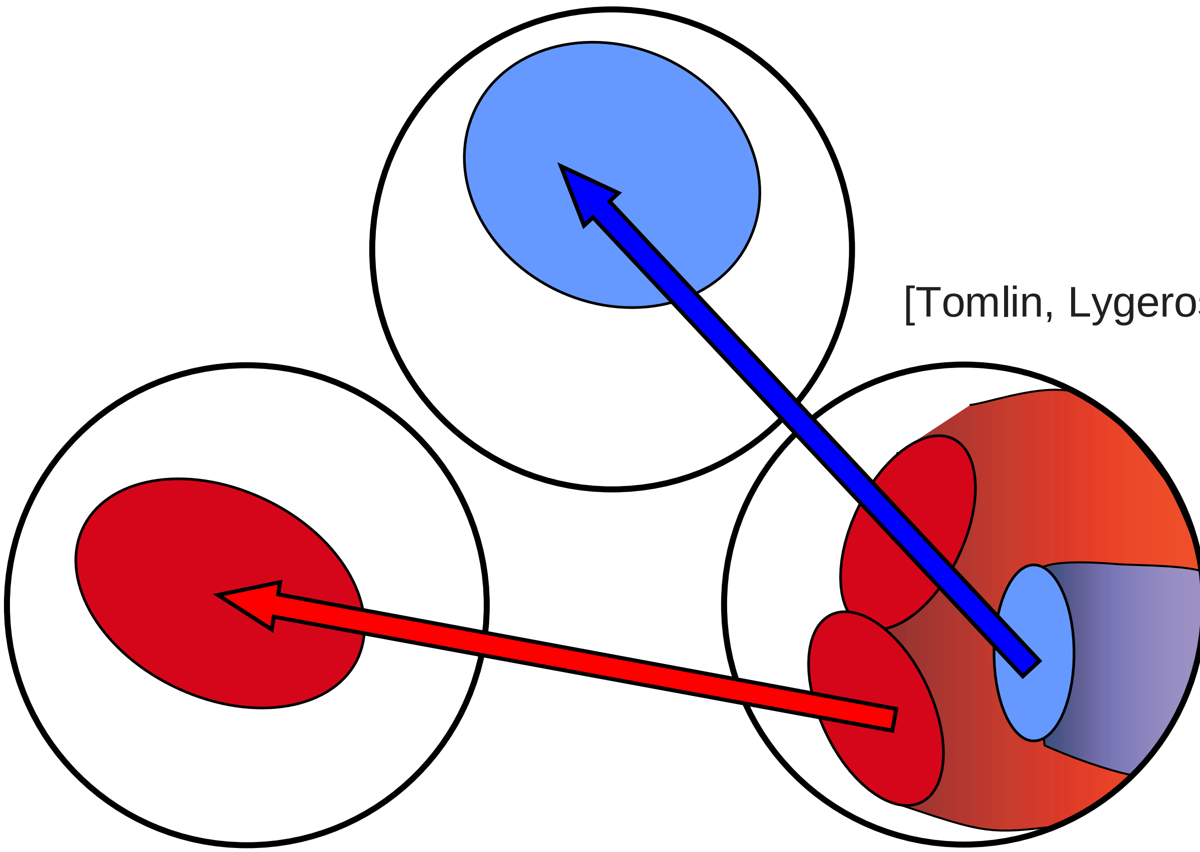
- Ability to choose maneuver start time further reduces unsafe set



[Tomlin, Mitchell & Ghosh, 2001]

Computing Hybrid Reachable Sets

- Compute continuous reachable set in each mode separately
  - Uncontrollable switches may introduce unsafe sets
  - Controllable switches may introduce safe sets
  - Forced switches introduce boundary conditions

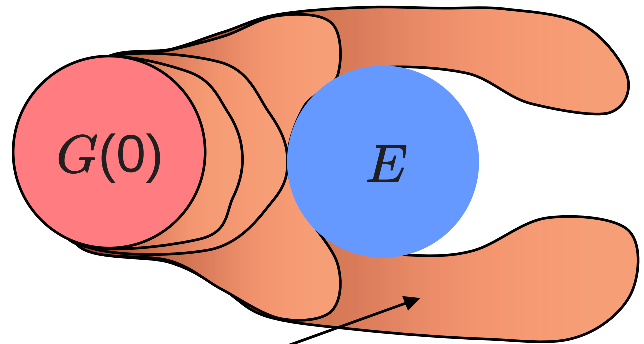


[Tomlin, Lygeros & Sastry, 2000]

Reach-Avoid Operator

- Compute set of states which reaches $G(0)$ without entering E

$$G(t) = \{x \in \mathbb{R}^n \mid \phi_G(x,t) \le 0\}$$
$$E = \{x \in \mathbb{R}^n \mid \phi_E(x) \le 0\}$$



Reach-Avoid Set $G(t)$

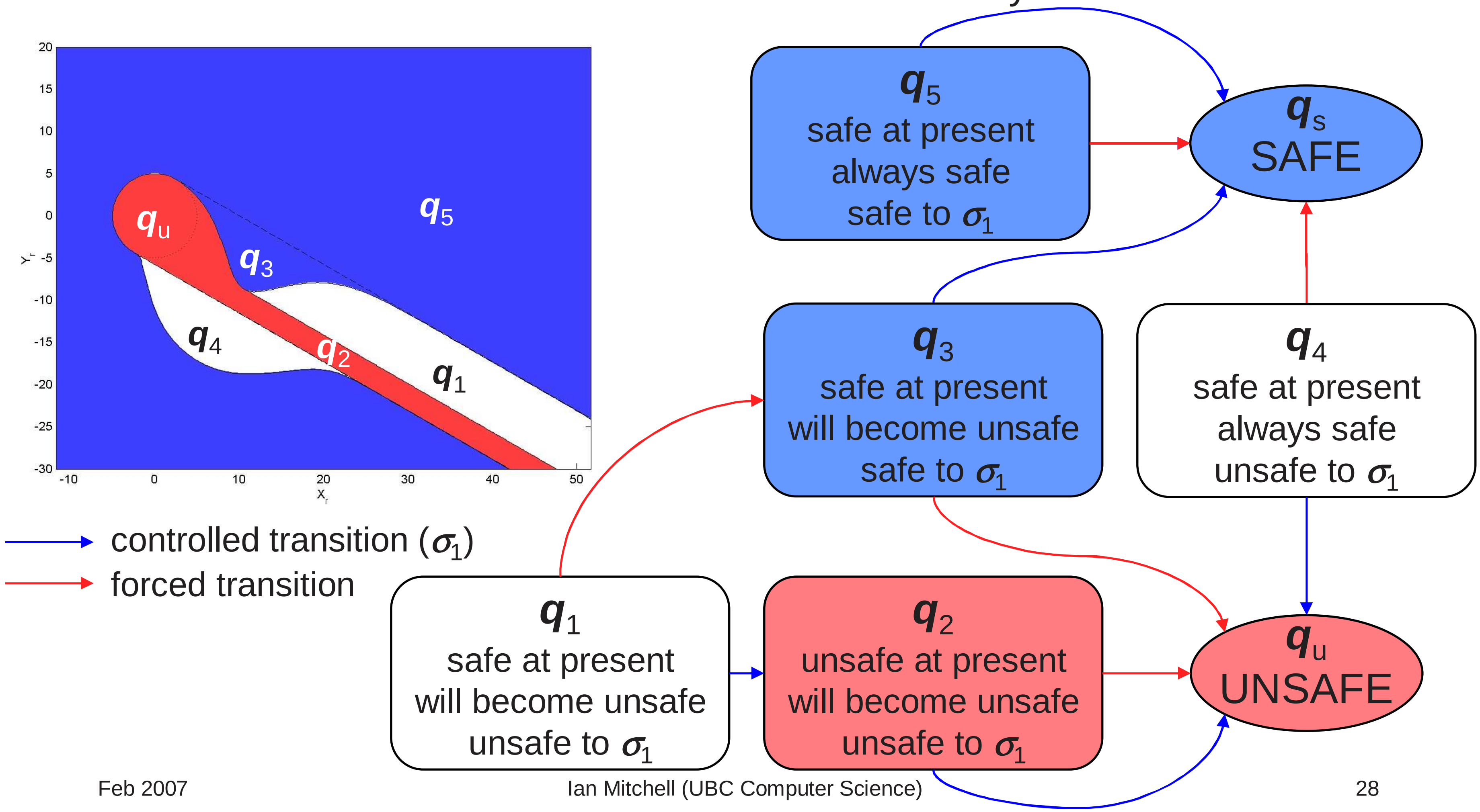
- Formulated as a constrained Hamilton-Jacobi equation or variational inequality
  - [Mitchell & Tomlin, 2000]

$$D_t\phi_G(x,t) + \min\left[0, H(x, D_x\phi_G(x,t))\right] = 0$$
$$\text{subject to: } \phi_G(x,t) \ge \phi_E(x)$$

- Level set can represent often odd shape of reach-avoid sets

Application: Discrete Abstractions

- It can be easier to analyze discrete automata than hybrid automata or continuous systems
  – Use reachable set information to abstract away continuous details



Application: Cockpit Display Analysis

- Controllable flight envelopes for landing and Take Off / Go Around (TOGA) maneuvers may not be the same
- Pilot’s cockpit display may not contain sufficient information to distinguish whether TOGA can be initiated