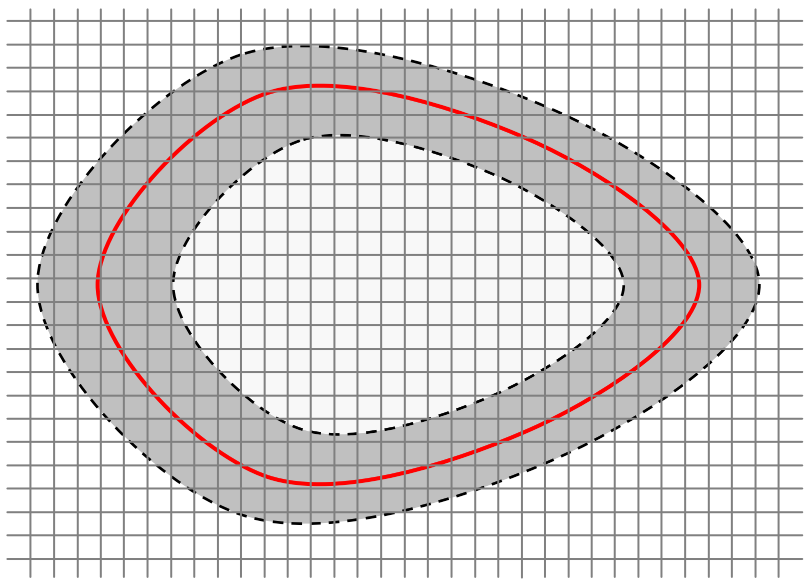
Efficiency

Narrowbanding / Local Level Set

Projections

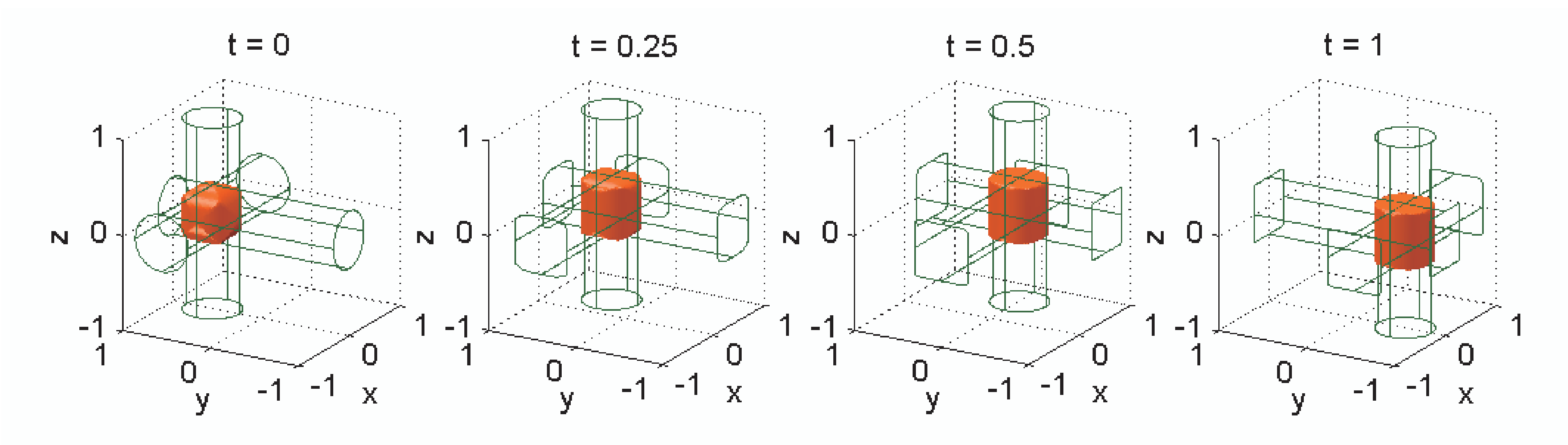
Reducing the Cost of Level Set Methods

- Solve Hamilton-Jacobi equation only in a band near interface
- Computational detail: handling stencils near edge of band
  - "Narrowbanding" uses low order accurate reconstruction whenever errors are detected
  - "Local level set" modifies Hamiltonian near edge of band
- Data structure detail: handling merging and breaking of interface



Projective Overapproximation

- Overapproximate reachable set of high dimensional system as the intersection of reachable sets for lower dimensional projections
  - [Mitchell & Tomlin, 2002]
  - Example: rotation of “sphere” about z-axis



Computing with Projections

- Forward and backward reachable sets for finite automata
  - Projecting into overlapping subsets of the variables, computing with BDDs [Govindaraju, Dill, Hu, Horowitz]
- Forward reachable sets for continuous systems
  - Projecting into 2D subspaces, representation by polygons [Greenstreet & Mitchell]
- Level set algorithms for geometric optics
  - Need multiple arrival time (viscosity solution gives first arrival time), so compute in higher dimensions and project down [Osher, Cheng, Kang, Shim & Tsai]

Hamilton-Jacobi in the Projection

- Consider x-z projection represented by level set $\phi_{xz}(x,z,t)$
  - Back projection into 3D yields a cylinder $\phi_{xz}(x,y,z,t)$
- Simple HJ PDE for this cylinder

$$D_t\phi_{xz}(x,y,z,t) + \sum_{i=1}^{3} p_i f_i(x,y,z) = 0 \quad \text{where} \begin{cases} p_1 & = D_x\phi_{xz}(x,y,z,t) \\ p_2 & = D_y\phi_{xz}(x,y,z,t) \\ p_3 & = D_z\phi_{xz}(x,y,z,t) \end{cases}$$

  - But for cylinder parallel to y-axis, $p_2 = 0$

$$D_t\phi_{xz}(x,y,z,t) + p_1 f_1(x,y,z) + p_3 f_3(x,y,z) = 0$$

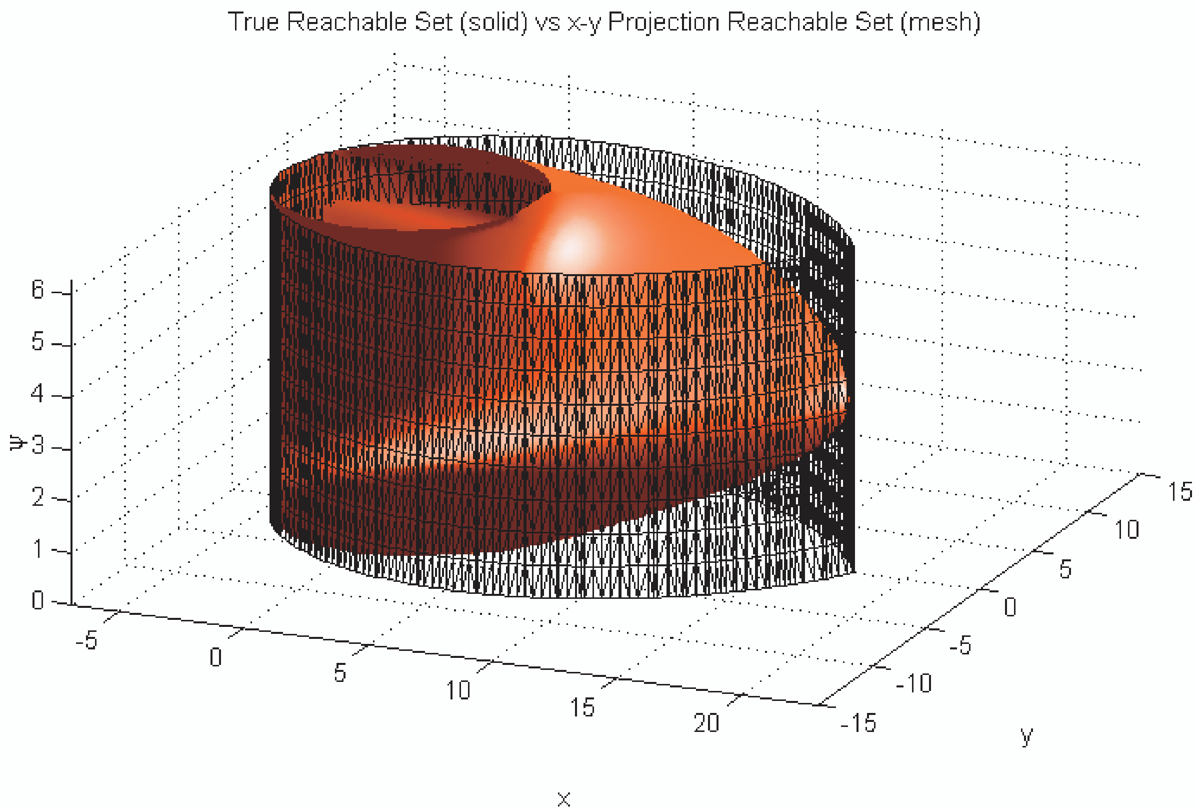
- What value to give free variable y in $f_i(x,y,z)$?
  - Treat it as a disturbance, bounded by the other projections

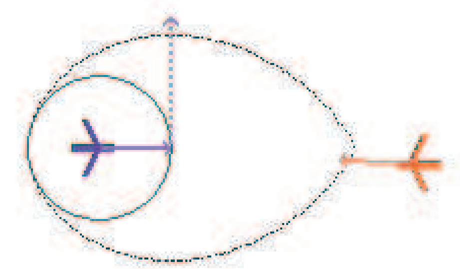
$$D_t\phi_{xz}(x,y,z,t) + \min_y \left[p_1 f_1(x,y,z) + p_3 f_3(x,y,z)\right] = 0$$

- Hamiltonian no longer depends on y, so computation can be done entirely in x-z space on $\phi_{xz}(x,z,t)$

Projective Collision Avoidance

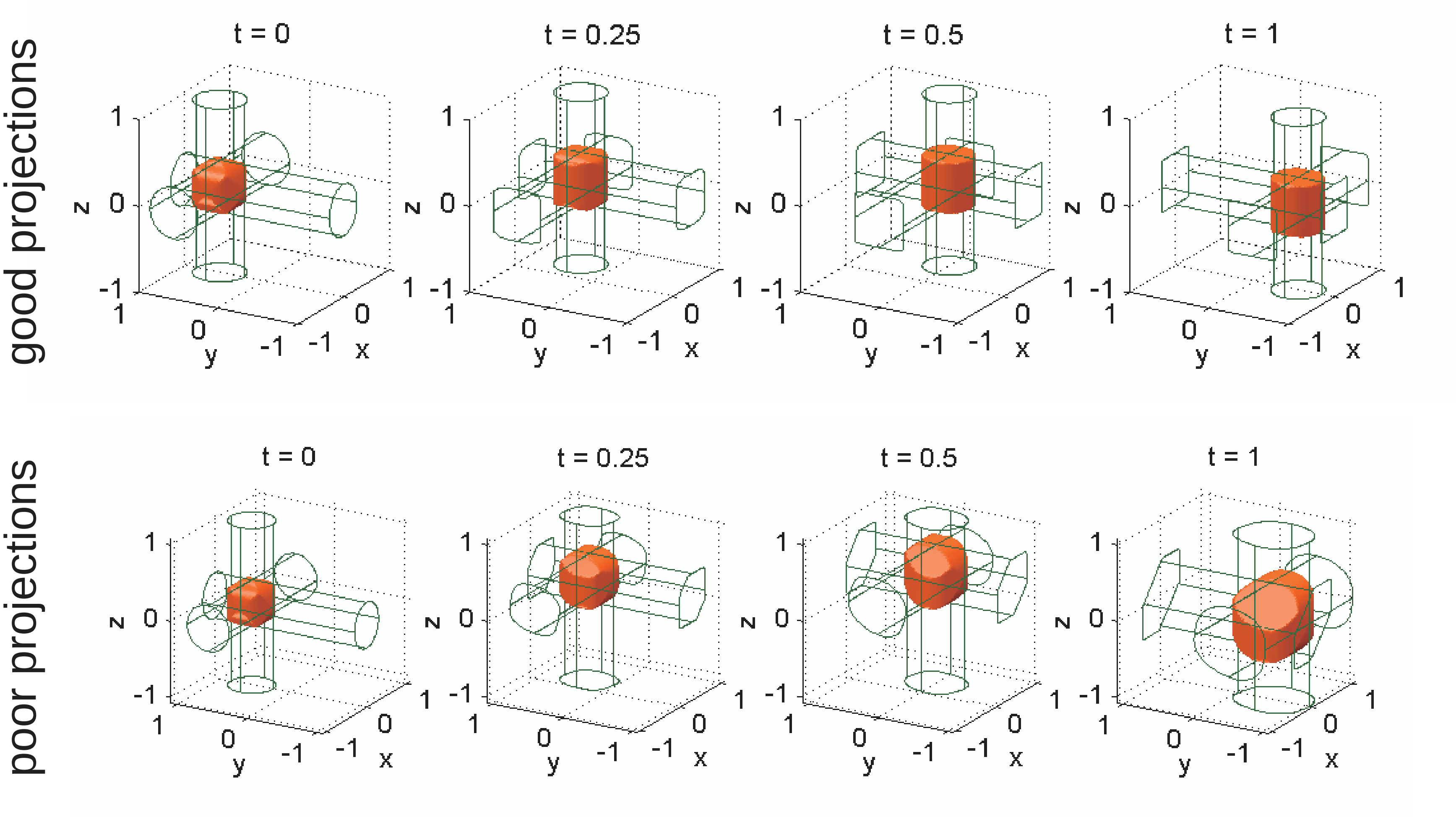
- Work strictly in relative x-y plane
  - Treat relative heading $\psi \in [\,0, 2\pi\,]$ as a disturbance input
  - Compute time: 40 seconds in 2D vs 20 minutes in 3D
  - Compare overapproximative prism (mesh) to true set (solid)





Projection Choices

- Poorly chosen projections may lead to large overapproximations
  - Projections need not be along coordinate axes
  - Number of projections is not constrained by number of dimensions



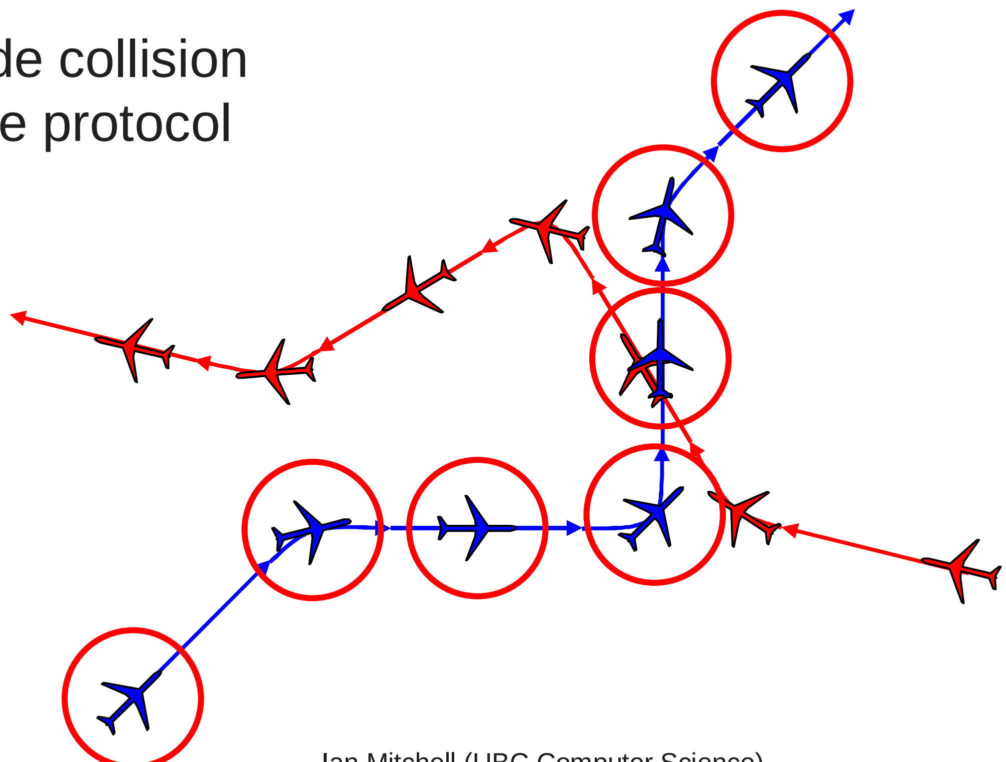
Hybrid System Reach Sets

Combining Continuous and Discrete Evolution

Why Hybrid Systems?

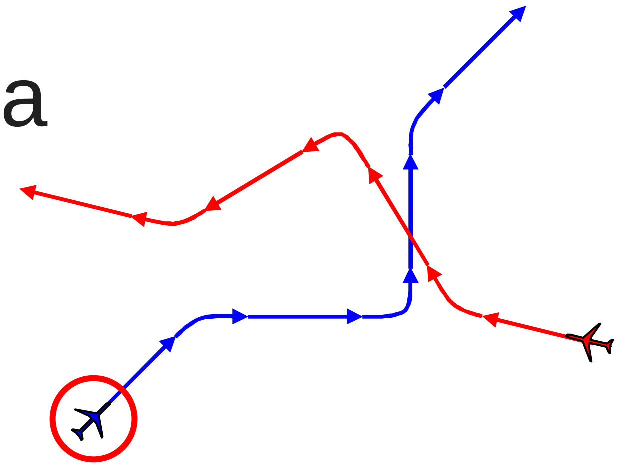
- Computers are increasingly interacting with external world
  - Flexibility of such combinations yields huge design space
  - Design methods and tools targeted (mostly) at either continuous or discrete systems
- Example: aircraft flight control systems

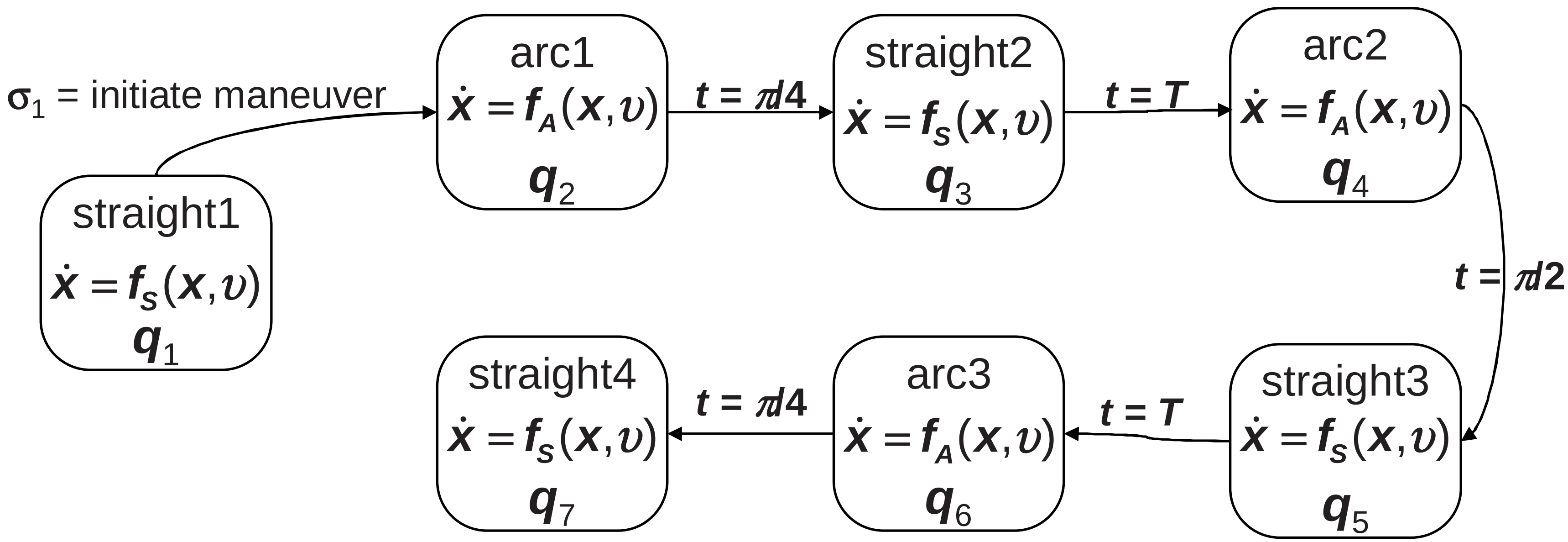
seven mode collision avoidance protocol



Hybrid Automata

- Discrete modes and transitions
- Continuous evolution within each mode





$$f_S\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -v + v\cos\psi \\ v\sin\psi \end{pmatrix}$$

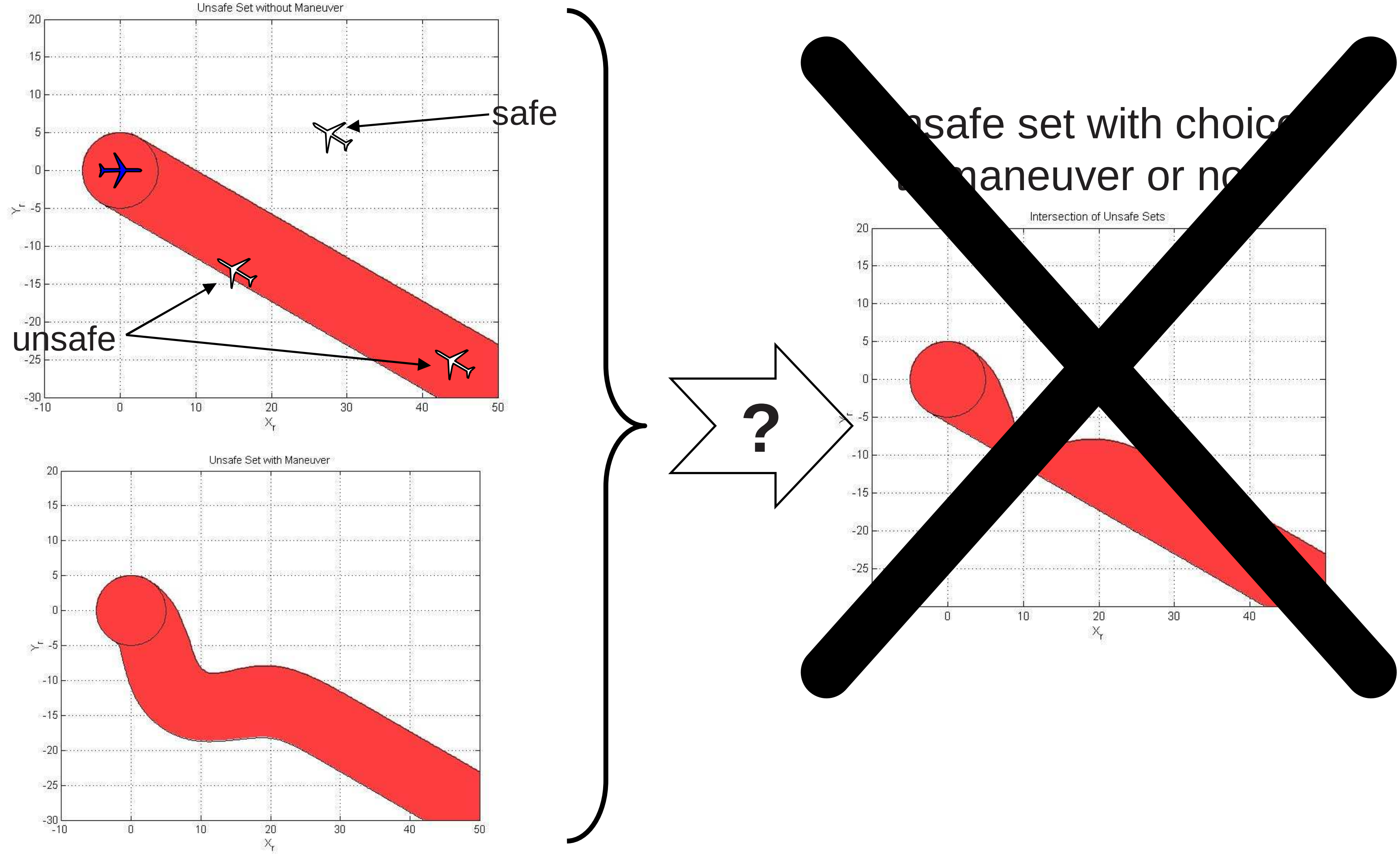
dynamics in straight modes

$$f_A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -v + v\cos\psi - x_2 \\ v\sin\psi + x_1 \end{pmatrix}$$

dynamics in arc modes

Seven Mode Safety Analysis

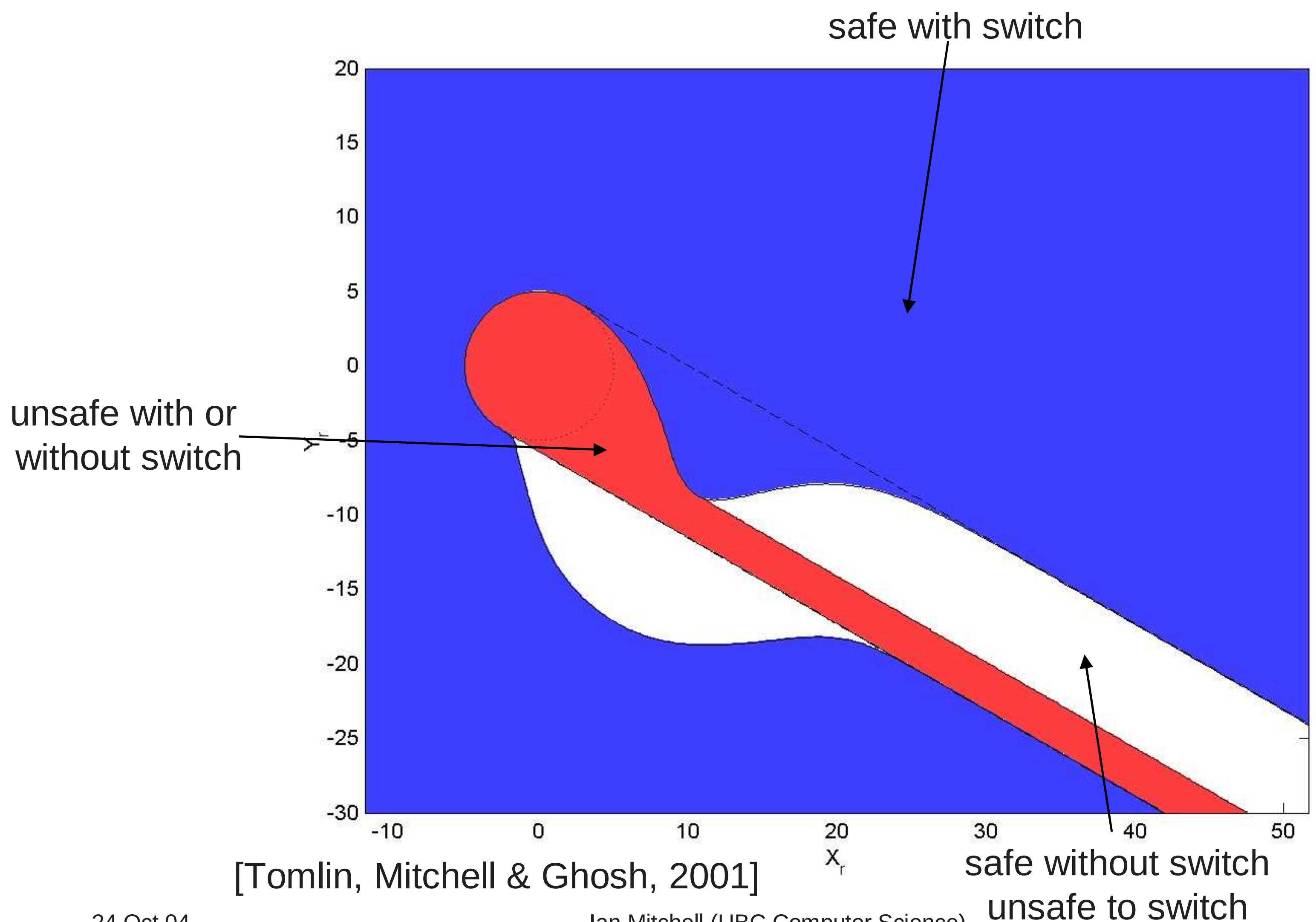
unsafe set without maneuver



unsafe set with maneuver

Seven Mode Safety Analysis

- Ability to choose maneuver start time further reduces unsafe set

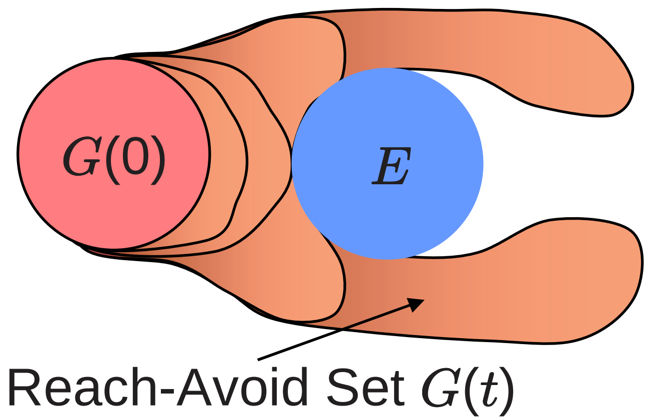


[Tomlin, Mitchell & Ghosh, 2001]

Reach-Avoid Operator

- Compute set of states which reaches $G(0)$ without entering E

$$G(t) = \{x \in \mathbb{R}^n \mid \phi_G(x,t) \leq 0\}$$
$$E = \{x \in \mathbb{R}^n \mid \phi_E(x) \leq 0\}$$



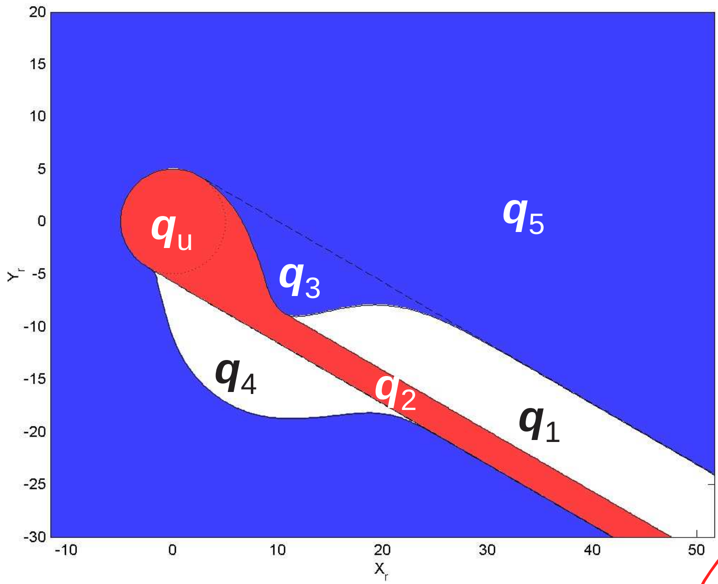
- Formulated as a constrained Hamilton-Jacobi equation or variational inequality
  - [Mitchell & Tomlin, 2000]

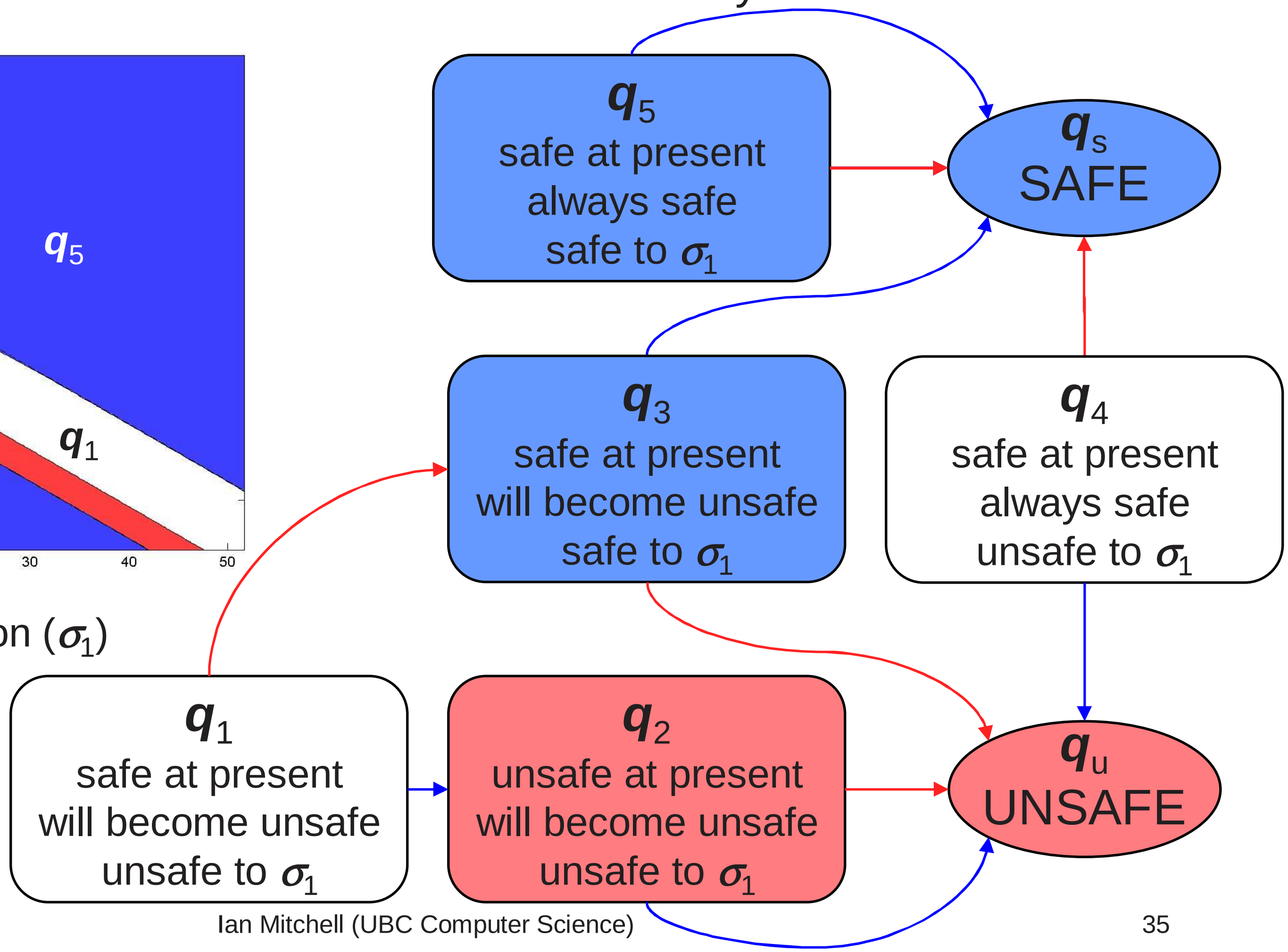
$$D_t\phi_G(x,t) + \min\left[0, H(x, D_x\phi_G(x,t))\right] = 0$$
$$\text{subject to: } \phi_G(x,t) \geq \phi_E(x)$$

- Level set can represent often odd shape of reach-avoid sets

Application: Discrete Abstractions

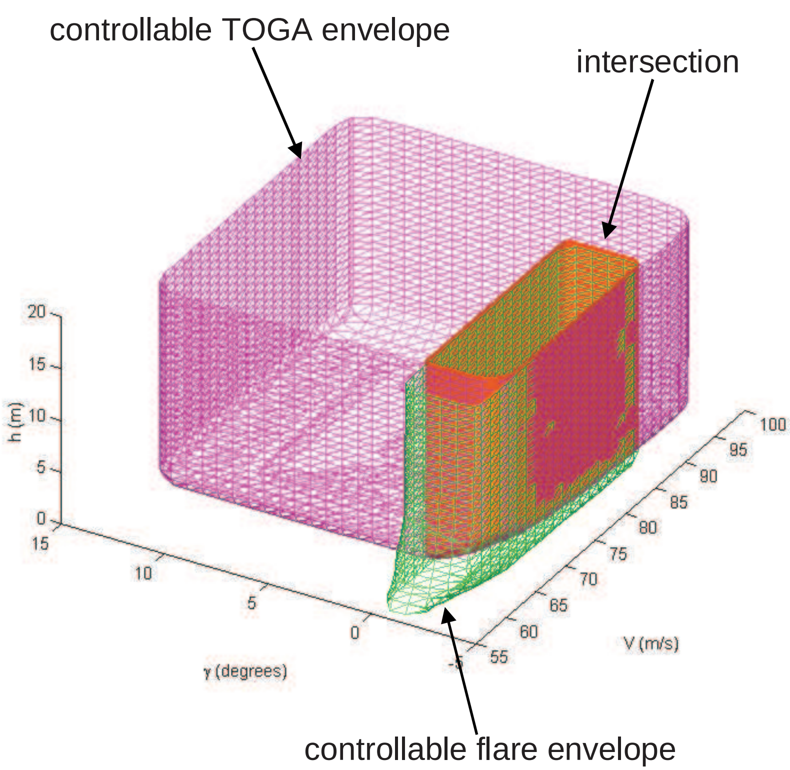
- It can be easier to analyze discrete automata than hybrid automata or continuous systems
  - Use reachable set information to abstract away continuous details

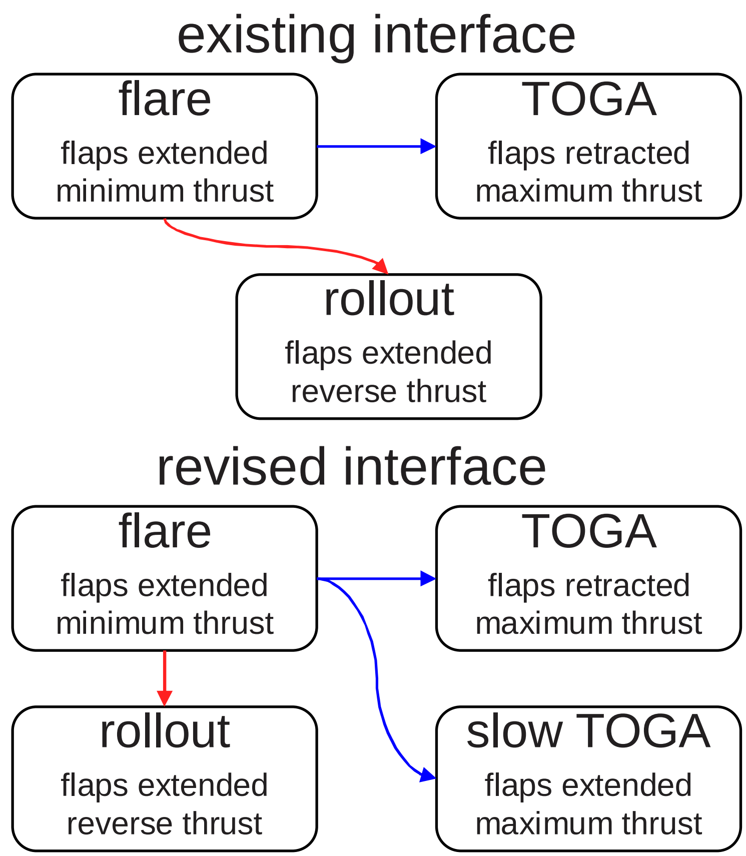




Application: Cockpit Display Analysis

- Controllable flight envelopes for landing and Take Off / Go Around (TOGA) maneuvers may not be the same
- Pilot's cockpit display may not contain sufficient information to distinguish whether TOGA can be initiated

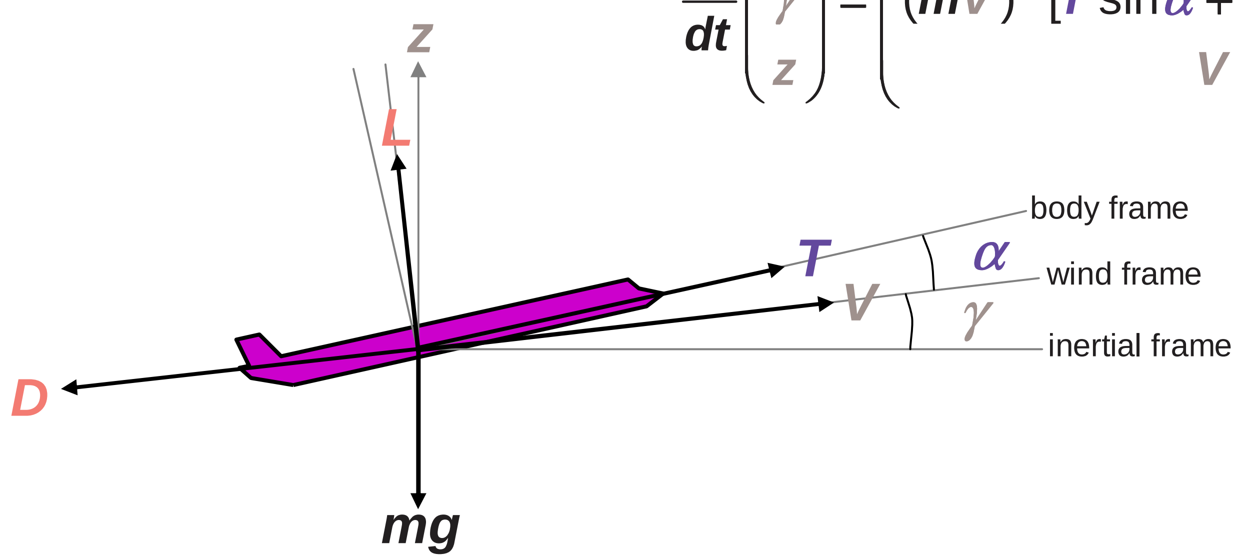




Application: Aircraft Autolander

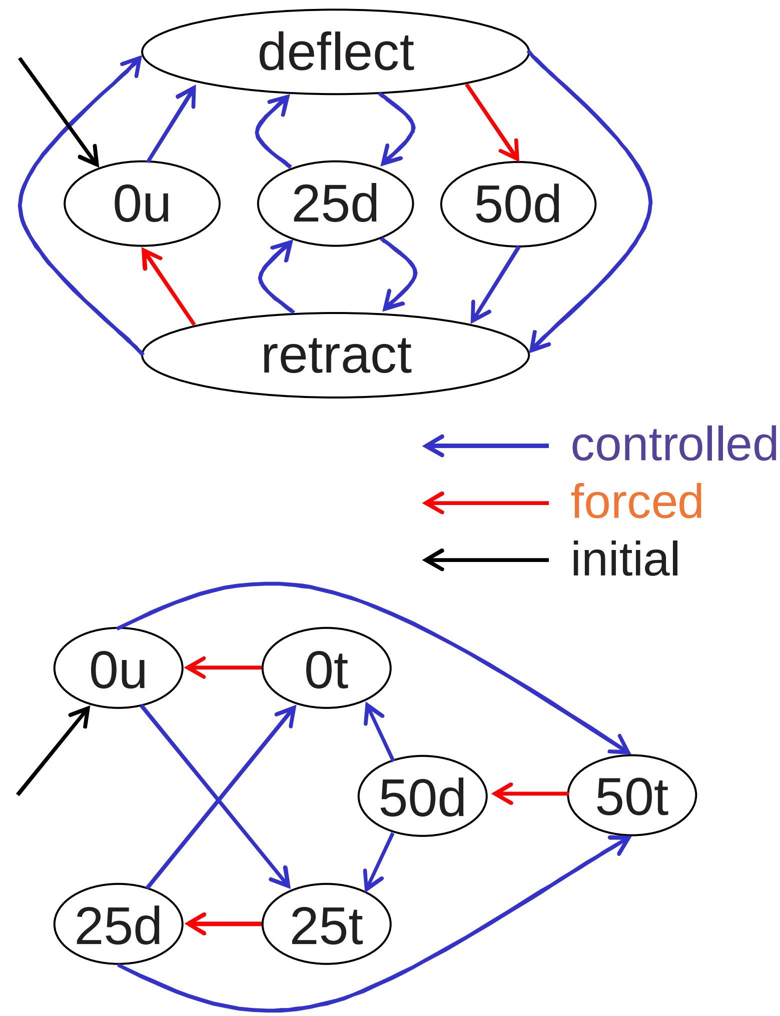
- Airplane must stay within safe flight envelope during landing
  - Bounds on velocity (V), flight path angle (γ), height (z)
  - Control over engine thrust (T), angle of attack (α), flap settings
  - Model flap settings as discrete modes of hybrid automata
  - Terms in continuous dynamics may depend on flap setting
  - [Mitchell, Bayen & Tomlin, 2001]

$$\frac{d}{dt}\begin{pmatrix} V \\ \gamma \\ z \end{pmatrix} = \begin{pmatrix} m^{-1}[T\cos\alpha - D(\alpha, V) - mg\sin\gamma] \\ (mV)^{-1}[T\sin\alpha + L(\alpha, V) - mg\cos\gamma] \\ V\sin\gamma \end{pmatrix}$$

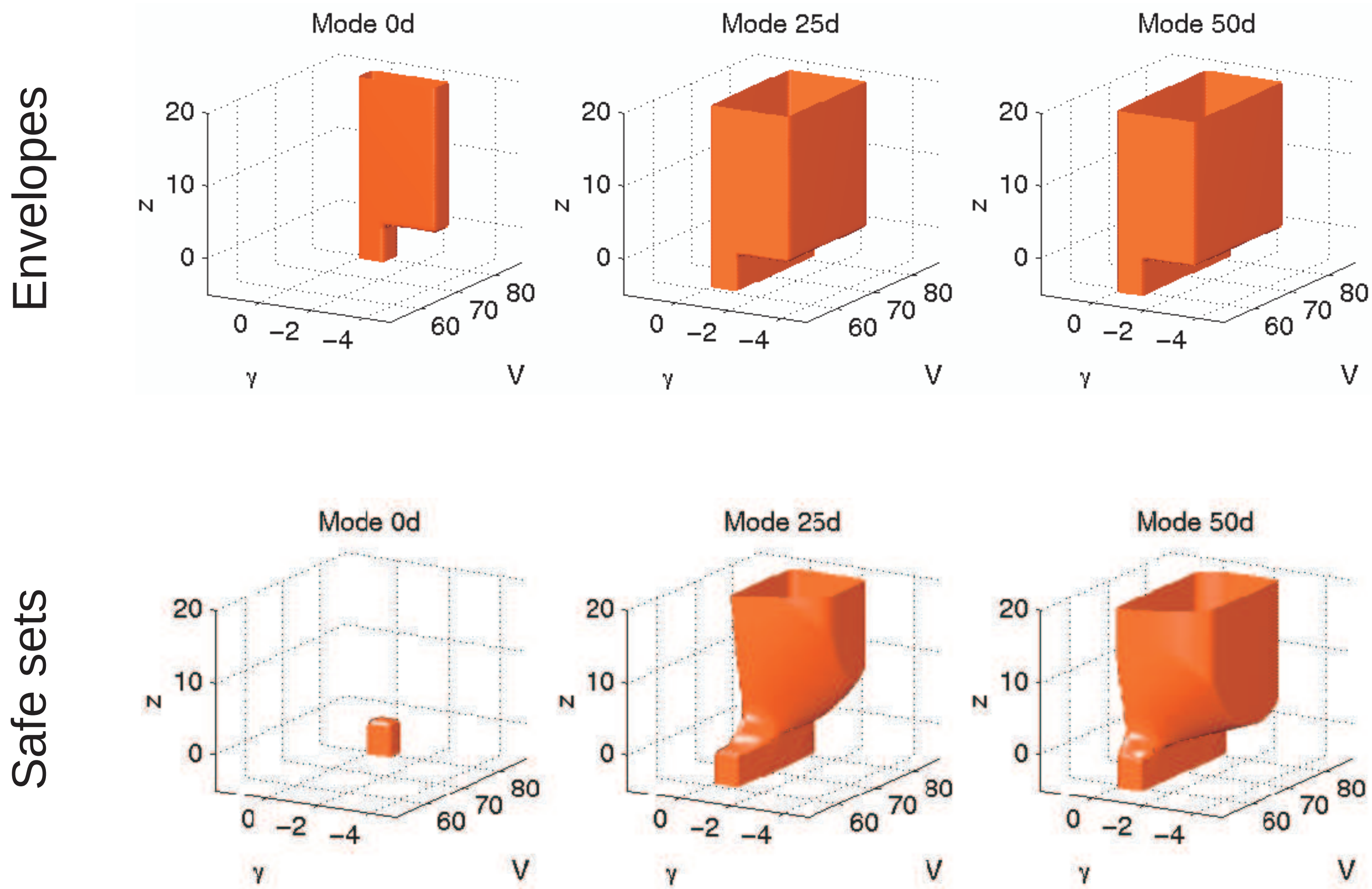


Landing Example: Discrete Model

- Flap dynamics version
  - Pilot can choose one of three flap deflections
  - Thirty seconds for zero to full deflection
- Implemented version
  - Instant switches between fixed deflections
  - Additional timed modes to remove Zeno behavior

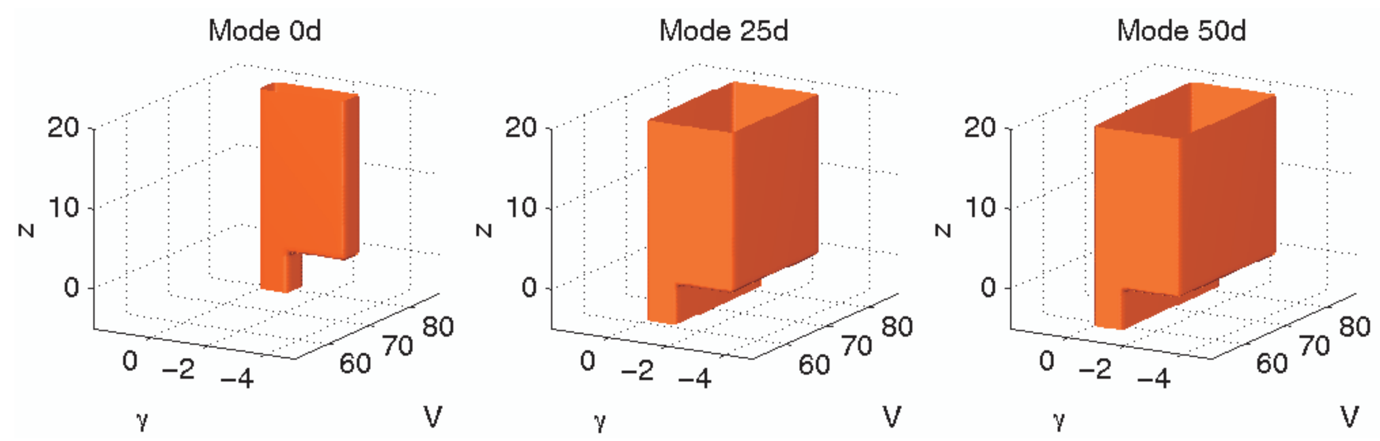


Landing Example: No Mode Switches

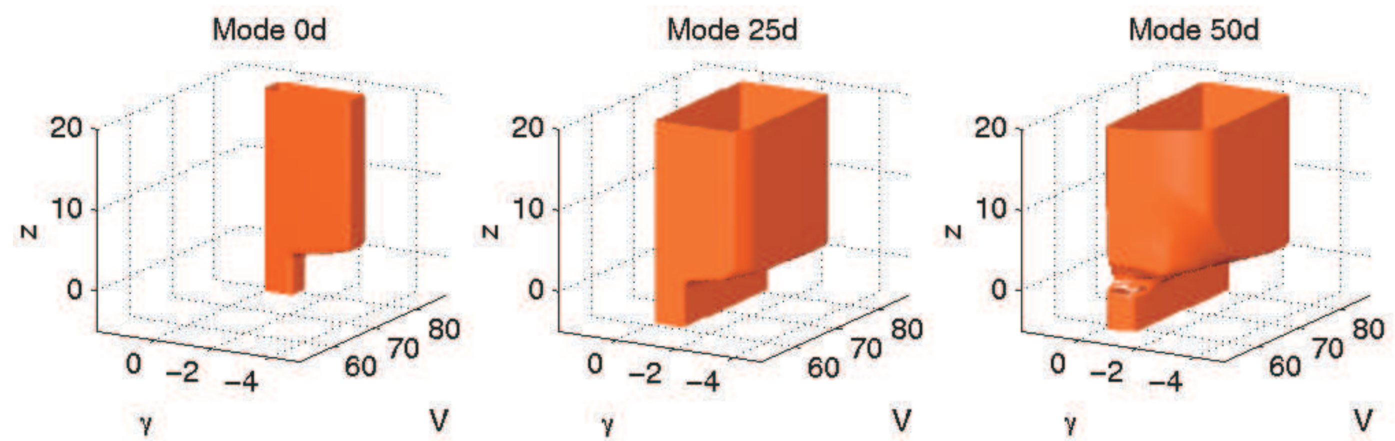


Landing Example: Mode Switches

Envelopes



Safe sets



Landing Example: Synthesizing Control

- For states at the boundary of the safe set, results of reach-avoid computation determine
  - What continuous inputs (if any) maintain safety
  - What discrete jumps (if any) are safe to perform
  - Level set values & gradients provide all relevant data