Eikonal Equation for Shortest Continuous Path and the Fast Marching Method

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Introduction

- Eikonal equation for shortest continuous path
  - Form of the stationary Hamilton Jacobi with certain constraints
  - Solution is value function V(x) which represents minimum cost to go from source x_s to any point x in continuous state space
  - Solution may be discontinuous -> Viscosity solutions
  - Numerous methods for solving equation

- Applications:
  - Path planning
  - Image segmentation (deformable models)

Stationary Hamilton-Jacobi

- Time dependent HJ (from last class):

$$\frac{\partial V(x,t)}{\partial t} + \min_{u \in U} \left[f(x,u) \cdot \nabla_X V(x,t) + g(x,u) \right] = 0$$

- Stationary HJ for path p(s)
  - Note that: $V(x,t) \rightarrow V(x)$
    $g(x,u) = c(x)$
  - Set x = p(s)
  - Define $f(x,u) = u(s) = \frac{dp(s)}{ds}$ to be direction of motion
  - Stationary HJ:

$$\min_{u} \left[\nabla V(x) \cdot u + c(x) \right] = 0$$

Eikonal Equation

- Solution V(x) represents optimum cost to go from current point x

- Place constraints on stationary HJ:
  - Path planning example: Path from x to target set T
  - Restrict to isotropic problem: $\|u\|_2 \leq 1$
  - or for some *p*-norm defined for $z \in \mathbb{R}^d$ by:

$$\|z\|_p = \left(\sum_{i=1}^{d} |z_i|^p \right)^{\frac{1}{p}}$$

  - Choose optimum control:

$$u(\cdot) = \frac{-\nabla V(x)}{\|\nabla V(x)\|}$$

Eikonal Equation

$$\min_{u}\left[\nabla V(x)\cdot u + c(x)\right] = 0$$

$$\frac{-\nabla V(x)\cdot\nabla V(x)}{\|\nabla V(x)\|} + c(x) = 0$$

- resulting in the Eikonal equation:

$$\|\nabla V(x)\|_2 = c(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \mathcal{T}$$

$$V(x) = 0 \quad \text{for } x \in \partial\mathcal{T}$$

- General form:

$$\|\nabla V(x)\|_{p*} = c(x)$$

$$V(x) = 0$$

Approximation of Value Function

- Thus optimum path found by gradient descent of value function:

$$\frac{dp}{ds} = \frac{\nabla V(x)}{\|\nabla V(x)\|}$$

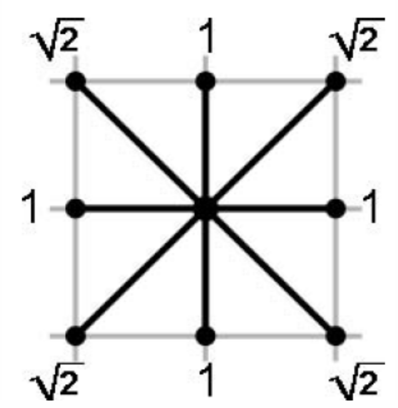
  - Value function will have no local minima
- But value function that solves Eikonal equation is rarely differentiable everywhere
- Viscosity solution:
  - Unique weak solution for V(x) exists that is bounded, differentiable
  - Finite difference approximation of V(x)
  - Found through Fast Marching Method

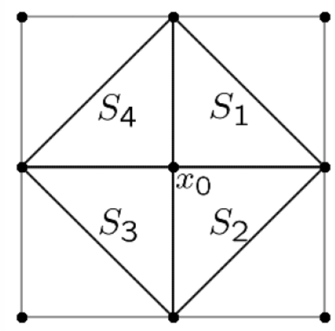
Overview of Fast Marching Method

- Fast marching method considered a ‘continuous Dijkstra’s method’
- Dijkstra's method keeps track of "current smallest cost" for reaching a grid point and fans out along the edges to touch the adjacent grid points.
- Value function for Dijkstra’s not defined on points in domain that are not nodes in the grid
- Action constrained to edges leading to neighbouring states
- Interpolation of actions to allow actions to non-grid nodes may not be optimal
- Solution: Use interpolation during construction of the value function
- FMM creates value function approximation, use gradient descent to find optimum path

Overview of Fast Marching Method

- Instead of nodes, use simplexes where optimum path may cross the face of the simplex





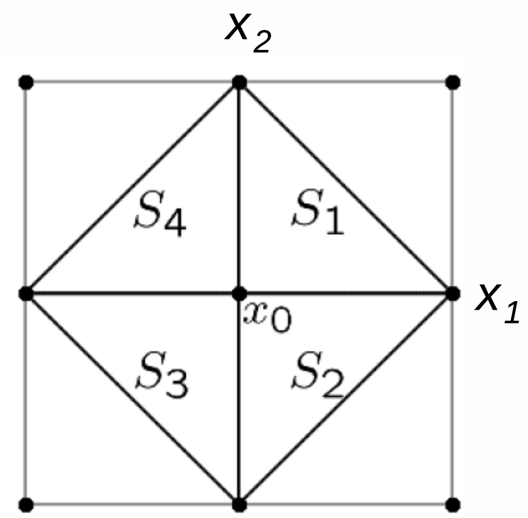
- General method:
  - Follows same general shortest path algorithm but with different update method than Dijkstra's
  - Plug finite difference approximation for $\nabla V(x)$ into Eikonal equation and solve for $V(x_0)$ in terms of $c(x_0)$ and $V(x_i)$ of neighbouring simplexes.

Approximation of Value Function

- Example: Specialized case for orthogonal grid of R^2:
- V_{12} is linear interpolant along edge

$$V(x_0) = \min_{\tilde{x} \in [x_1, x_2]} \left(V_{12}(\tilde{x}) + c(x_0) \|\tilde{x} - x_0\|_2 \right)$$

$$\left\| \nabla V(x_0) \right\|_2 = \sqrt{\left(\frac{V_0 - V_1}{\Delta x} \right)^2 + \left(\frac{V_0 - V_2}{\Delta x} \right)^2} = c(x_0)$$



- Solve for V_o for S1 simplex (quadratic solution):

$$(2V_0 - V_1 - V_2)^2 = 2\Delta x^2 c(x_0)^2 - (V_1 - V_2)^2$$

$$V^{(S1)}{}_0 = \frac{1}{2}\left(V_1 + V_2 + \sqrt{2\Delta x^2 c(x_0)^2 - (V_1 - V_2)^2} \right)$$

FMM for Shortest Path

- Standard shortest path method:

$$\begin{array}{l}
\textbf{foreach } x_i \in \mathcal{G}_n \setminus \mathcal{T} \textbf{ do } V(x_i) = +\infty \\
\textbf{foreach } x_i \in \mathcal{T} \textbf{ do } V(x_i) = 0 \\
\mathcal{Q} \leftarrow \mathcal{G}_n \\
\textbf{while } \mathcal{Q} \neq \emptyset \textbf{ do} \\
\quad x_i \leftarrow \texttt{ExtractMin}(\mathcal{Q}) \\
\quad \textbf{foreach } x_j \in \mathcal{N}_n(x_i) \textbf{ do} \\
\quad\quad V(x_j) \leftarrow \texttt{Update}(x_j, \mathcal{N}(x_j), V, c)
\end{array}$$

- Dijkstra's update: $\texttt{Update}(x_j, \mathcal{N}(x_j), V, c) = c(x_j) + \min_{x_k \in \mathcal{N}_n(x_j)} V(x_k)$
- Fast marching method update:

$$\begin{array}{l}
\textbf{Input}: x_0, \mathcal{N}(x_0), V, c \\
\textbf{Output}: V(x_0) \\
\textbf{foreach } S \in \mathcal{N}_s(x_0) \textbf{ do} \\
\quad \text{Compute } V^{(S)}(x_0) \text{ from} \\
\textbf{return } \min_S V^{(S)}(x_0)
\end{array}$$

$$V_0|_{p*=2} = \frac{1}{2}\left(V_1 + V_2 + \sqrt{2\Delta x^2 c(x_0)^2 - (V_1 - V_2)^2}\right)$$

General Approximation

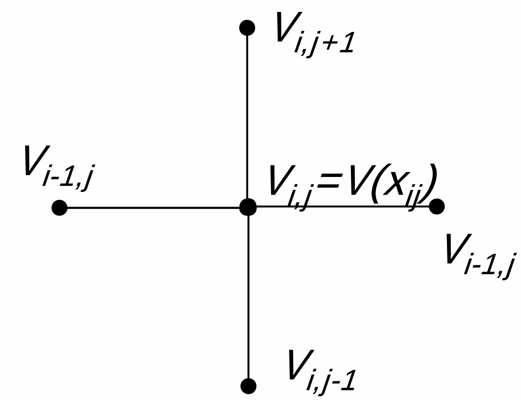
- Finite difference approximation (Sethian 1996):

$$\left\|\nabla V(x_{ij})\right\| = \sqrt{\max\left(D_{ij}^{-x}V, -D_{ij}^{+x}V, 0\right)^2 + \max\left(D_{ij}^{-y}V, -D_{ij}^{+y}V, 0\right)^2}$$

- where

$$D_{ij}^{-x}V = \frac{V_{i,j} - V_{i-1,j}}{\Delta x} \qquad D_{ij}^{-y}V = \frac{V_{i,j} - V_{i,j-1}}{\Delta x}$$

$$D_{ij}^{+x}V = \frac{V_{i+1,j} - V_{i,j}}{\Delta x} \qquad D_{ij}^{+y}V = \frac{V_{i,j+1} - V_{i,j}}{\Delta x}$$



- Produces quadratic equation to solve for $V(x_{ij})$
- Upwind different structure allows information to propagate from smaller V values to larger values

FMM for Different Norms

- Different norms can be used by modifying finite difference approximation:

$$||\nabla V_0||_1 = \frac{1}{\Delta x}(|V_0 - V_1| + |V_0 - V_2|),$$
$$||\nabla V_0||_\infty = \frac{1}{\Delta x}\max(|V_0 - V_1|, |V_0 - V_2|)$$

  - Resulting solutions for V_0:

$$V_0|_{p*=1} = \frac{1}{2}(\Delta x c(x_0) + V_1 + V_2),$$
$$V_0|_{p*=\infty} = \Delta x c(x_0) + \min(V_1, V_2)$$

References

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Questions?