Decision Theory: VE for Decision Networks, Sequential Decisions, Optimal Policies for Sequential Decisions

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UBC CS 322 - Decision Theory 3

April 3, 2013

Textbook §9.2.1, 9.3

# Announcements (1)

- Assignment 4 was due today.
- The list of short questions for the final is online ... please use it!
- Please submit suggested review topics on Connect for review lecture(s).
- Previous final has been posted.
- Additional review lecture(s) and TA hours will be scheduled before the final, if needed.
- TA hours to continue as scheduled during exam period, unless as posted otherwise to Connect.
- Exercise 12, for single-stage Decision Networks, and Exercise 13, for multi-stage Decision Networks, have been posted on the home page along with Alspace auxiliary files.

# Announcements (2)

- Teaching Evaluations are online
  - You should have received a message about them
  - Secure, confidential, mobile access
- Your feedback is important!
  - Allows us to assess and improve the course material
  - I use it to assess and improve my teaching methods
  - The department as a whole uses it to shape the curriculum
  - Teaching evaluation results are important for instructors
    - · Appointment, reappointment, tenure, promotion and merit, salary
  - UBC takes them very seriously (now)
  - Evaluations close at 11:59PM on April 9, 2013.
    - Before exam, but instructors can't see results until *after* we submit grades
  - Please do it!
- Take a few minutes and visit <u>https://eval.olt.ubc.ca/science</u>

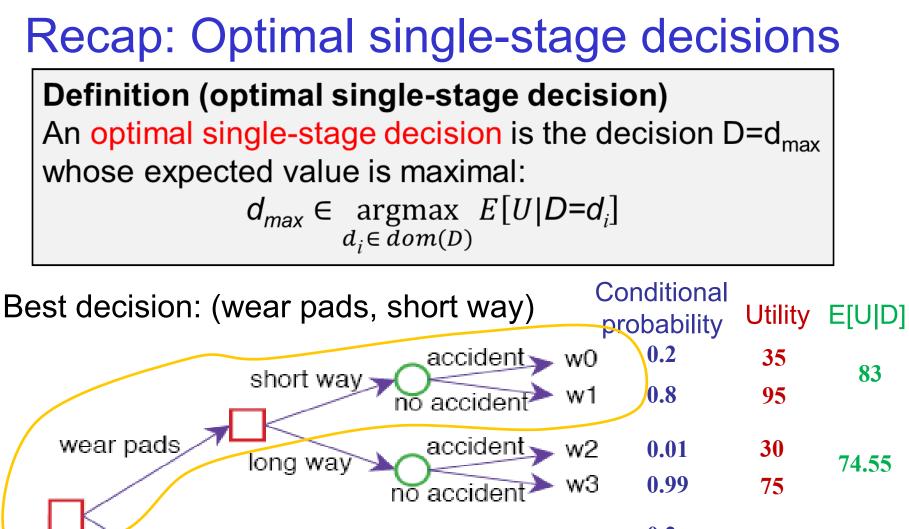
# Lecture Overview

Recap: Single-Stage Decision Problems

- Single-Stage decision networks
- Variable elimination (VE) for computing the optimal decision
- Sequential Decision Problems
  - General decision networks
  - Policies
- Expected Utility and Optimality of Policies
- Computing the Optimal Policy by Variable Elimination
- Summary & Perspectives

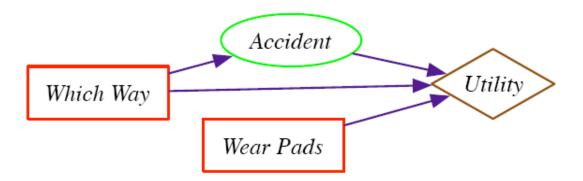
### **Recap: Single vs. Sequential Actions**

- Single Action (aka One-Off Decisions)
  - One or more primitive decisions that can be treated as a single macro decision to be made before acting
- Sequence of Actions (Sequential Decisions)
  - Repeat:
    - observe
    - act
  - Agent has to take actions not knowing what the future brings



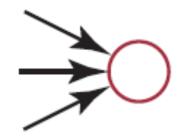


# Recap: Single-Stage decision networks

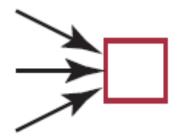


- Compact and explicit representation
  - Compact: each random/decision variable only occurs once
  - Explicit: dependences are made explicit
    - e.g., which variables affect the probability of an accident?
- Extension of Bayesian networks with
  - Decision variables
  - A single utility node

## Recap: Types of nodes in decision networks



- A random variable is drawn as an ellipse.
  - Parents pa(X): encode dependence
     Conditional probability p( X | pa(X) )
     Random variable X is conditionally independent
     of its non-descendants given its parents
  - Domain: the values it can take at random



- A decision variable is drawn as an rectangle.
  - Parents pa(D)
    - information available when decision D is made
      - Single-stage: pa(D) only includes decision variables
  - Domain: the values the agents can choose (actions)

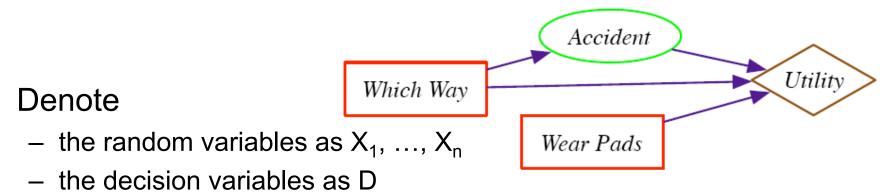


- A utility node is drawn as a diamond.
  - Parents pa(U): variables utility directly depends on
    - utility U( pa(U) ) for each instantiation of its parents
  - Domain: does not have a domain!

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#### Computing the optimal decision: we can use VE

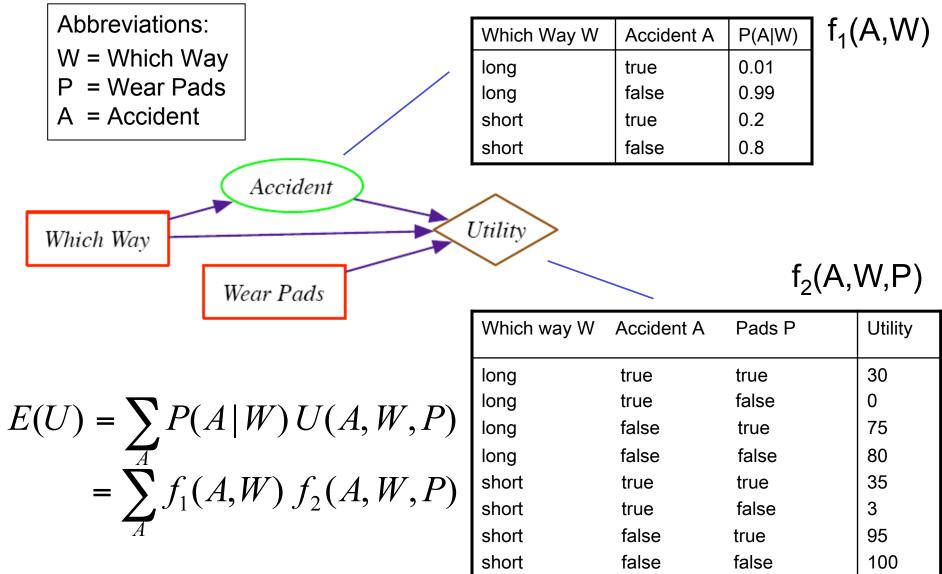


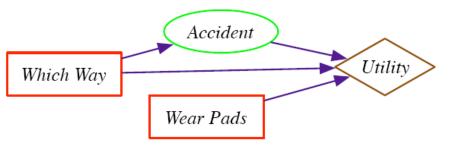
the parents of node N as pa(N)

$$E(U) = \sum_{X_1, \dots, X_n} P(X_1, \dots, X_n \mid D) U(pa(U))$$
  
= 
$$\sum_{X_1, \dots, X_n} \prod_{i=1}^n P(X_i \mid pa(X_i)) U(pa(U))$$

- To find the optimal decision we can use VE:
  - 1. Create a factor for each conditional probability and for the utility
  - 2. Sum out all random variables, one at a time
    - This creates a factor on D that gives the expected utility for each d<sub>i</sub>
  - 3. Choose the d<sub>i</sub> with the maximum value in the factor

### VE Example: Step 1, create initial factors

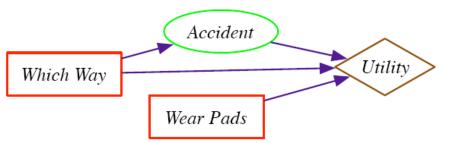




Step 2a: compute product  $f_1(A,W) \times f_2(A,W,P)$ 

#### What is the right form for the product $f_1(A,W) \times f_2(A,W,P)$ ?

**f(A,W) f(A,P) f(A) f(A,P,W)** 



Step 2a: compute product  $f(A,W,P) = f_1(A,W) \times f_2(A,W,P)$ 

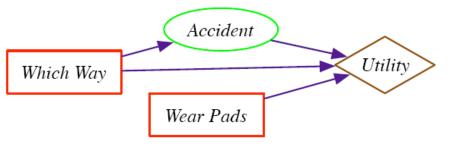
What is the right form for the product  $f_1(A,W) \times f_2(A,W,P)$ ? •It is f(A,P,W):

the domain of the product is the union of the multiplicands' domains • $f(A,P,W) = f_1(A,W) \times f_2(A,W,P)$ 

- I.e.,  $f(A=a,P=p,W=w) = f_1(A=a,W=w) \times f_2(A=a,W=w,P=p)$ 

Which Way	Accider	nt	Utility		•	•	oute prod (,W) × f <sub>2</sub> (	
	Wear Pad	5	f(A=a,F	P=p,₩	$V=w) = f_1(A)$	=a,W=w)	× f <sub>2</sub> (A=a,W=	=w,P=p)
Which way W	Accident A	f <sub>1</sub> (A,W)						
long	true	0.01			Which way W	Accident A	Pads P	f(A,W,P)
long	false	0.99			long	true	true	0.01 * 30
short	true	0.2			long	true	false	0.01 00
short	false	0.8			long	false	true	
Which way W	Accident A	Pads P	f <sub>2</sub> (A,W,P)	]	long short	false true	false true	???
long	true	true	30		short	true	false	
long	true	false	0		short	false	true	
long	false	true	75		short	false	false	
long	false	false	80					ļ
short	true	true	35					
short	true	false	3		0.99 *	30 (	).01 * 80	
short	false	true	95					
short	false	false	100		0.99	* 80 (	).8 * 30	14

Which Way	Accider	nt	Utili	ty		•	•	ute prod ,W) × f <sub>2</sub> (	
	Wear Pad.	5	f(A	=a,F	P=p,V	$V=w) = f_1(A)$	.=a,W=w) >	× f <sub>2</sub> (A=a,W	=w,P=p)
Which way W	Accident A	f <sub>1</sub> (A,W)							
long	true	0.01				Which way W	Accident A	Pads P	f(A,W,P)
long	false	0.99				long	true	true	0.01 * 30
short	true	0.2				long	true	false	0.01*0
short	false	0.8				long	false	true	0.99*75
Which way W	Accident A	Pads P	f <sub>2</sub> (A,V	V,P)	1	long	false	false	0.99*80
						short	true	true	0.2*35
long	true	true	30			short	true	false	0.2*3
long	true	false	0			short	false	true	0.8*95
long	false	true	75			short	false	false	0.8*100
long	false	false	80						
short	true	true	35						
short	true	false	3						
short	false	true	95						
short	false	false	100						15



Step 2b: sum A out of the product f(A,W,P):

$$f_3(W,P) = \sum_A f(A,W,P)$$

Which way W	Pads P	f <sub>3</sub> (W,P)
long long	true false	0.01*30+0.99*75=74.55
short	true	??
short	false	

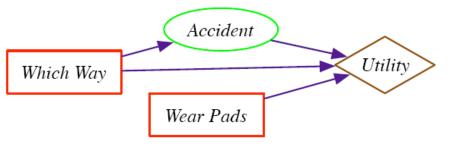
0.2\*35 + 0.2\*0.3

0.2\*35 + 0.8\*95

0.99\*80 + 0.8\*95

0.8 \* 95 + 0.8\*100

Which way W	Accident A	Pads P	f(A,W,P)
long	true	true	0.01 * 30
long	true	false	0.01*0
long	false	true	0.99*75
long	false	false	0.99*80
short	true	true	0.2*35
short	true	false	0.2*3
short	false	true	0.8*95
short	false	false	0.8*100



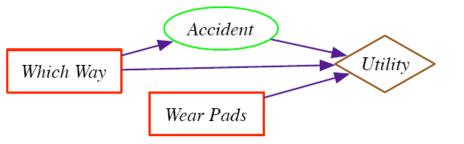
Step 2b: sum A out of the product f(A,W,P):

$$f_3(W,P) = \sum_A f(A,W,P)$$

Which way W	Pads P	f <sub>3</sub> (W,P)
long	true	0.01*30+0.99*75=74.55
long	false	0.01*0+0.99*80=79.2
short	true	0.2*35+0.8*95=83
short	false	0.2*3+0.8*100=80.6

Which way W	Accident A	Pads P	f(A,W,P)
long	true	true	0.01 * 30
long	true	false	0.01*0
long	false	true	0.99*75
long	false	false	0.99*80
short	true	true	0.2*35
short	true	false	0.2*3
short	false	true	0.8*95
short	false	false	0.8*100

### VE example: step 3, choose decision with max E(U)



Step 2b: sum A out of the product f(A,W,P):

$$f_3(W,P) = \sum_A f(A,W,P)$$

Which way W	Pads P	f <sub>3</sub> (W,P)
long	true	0.01*30+0.99*75=74.55
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Which way W	Accident A	Pads P	f(A,W,P)
long	true	true	0.01 * 30
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long	false	false	0.99*80
short	true	true	0.2*35
short	true	false	0.2*3
short	false	true	0.8*95
short	false	false	0.8*100

# The final factor encodes the expected utility of each decision



Thus, taking the short way but wearing pads is the best choice, with an expected utility of 83

# Variable Elimination for Single-Stage Decision Networks: Summary

- 1. Create a factor for each conditional probability and for the utility
- 2. Sum out all random variables, one at a time
  - This creates a factor on D that gives the expected utility for each d<sub>i</sub>
- 3. Choose the d<sub>i</sub> with the maximum value in the factor

This is Algorithm OptimizeSSDN, in P&M, Section 9.2.1, p.387

# Learning Goals So Far For Decisions

- Compare and contrast stochastic single-stage (one-off) decisions vs. multistage (sequential) decisions
- Define a Utility Function on possible worlds
- Define and compute optimal one-off decisions
- Represent one-off decisions as single stage decision networks
- Compute optimal decisions by Variable Elimination

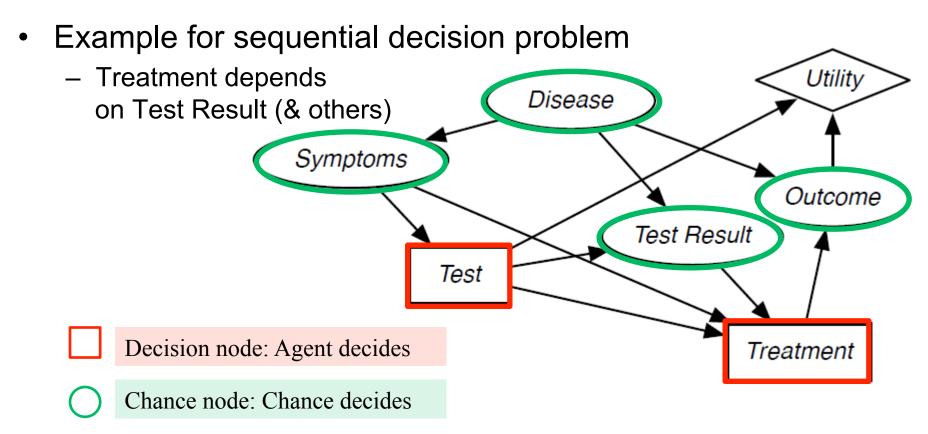
# Lecture Overview

- Recap: Single-Stage Decision Problems
  - Single-Stage decision networks
  - Variable elimination (VE) for computing the optimal decision
  - **Sequential Decision Problems** 
    - General decision networks
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# **Sequential Decision Problems**

- An intelligent agent doesn't make a multi-step decision and carry it out blindly
  - It would take new observations it makes into account
- A more typical scenario:
  - The agent observes, acts, observes, acts, ...
- Subsequent actions can depend on what is observed
  - What is observed often depends on previous actions
  - Often the sole reason for carrying out an action is to provide information for future actions
    - For example: diagnostic tests, spying
- General Decision networks:
  - Just like single-stage decision networks, with one exception: the parents of decision nodes can include random variables

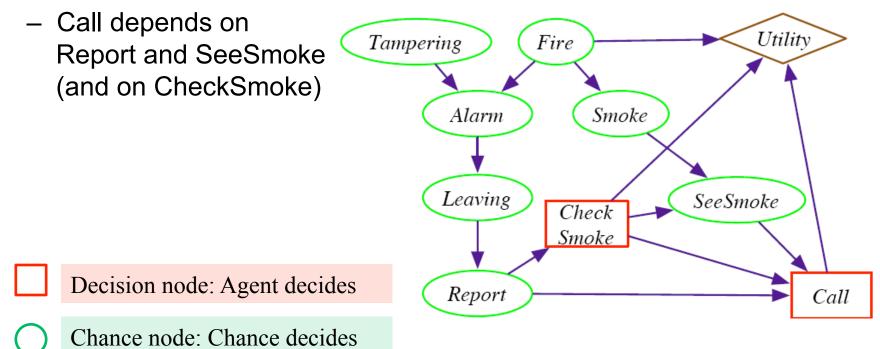
### Sequential Decision Problems: Example



- Each decision D<sub>i</sub> has an information set of variables pa(D<sub>i</sub>), whose value will be known at the time decision D<sub>i</sub> is made
  - pa(Test) = {Symptoms}
  - pa(Treatment) = {Test, Symptoms, TestResult}

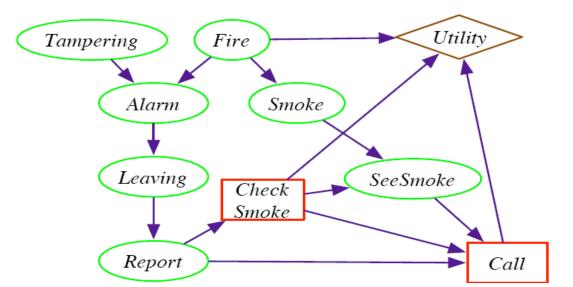
### Sequential Decision Problems: Example

• Another example for sequential decision problems



# **Sequential Decision Problems**

- What should an agent do?
  - What an agent should do depends on what it will do in the future
    - E.g. agent only needs to check for smoke if that will affect whether it calls
  - What an agent does in the future depends on what it did before
    - E.g. when making the decision it needs to know whether it checked for smoke
  - We will get around this problem as follows
    - The agent has a conditional plan of what it will do in the future
    - We will formalize this conditional plan as a policy



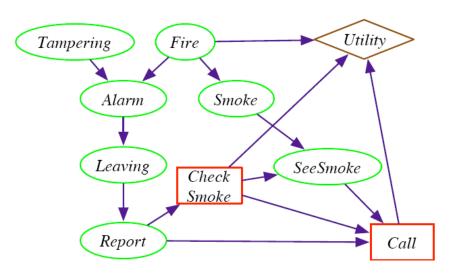
#### **Policies for Sequential Decision Problems**

Definition (Policy)

A policy is a sequence of  $\delta_1, \ldots, \delta_n$  decision functions

 $\delta_i$ : dom( $pa(D_i)$ )  $\rightarrow$  dom( $D_i$ )

This policy means that when the agent has observed  $o \in \text{dom}(pa(D_i))$ , it will do  $\delta_i(o)$ 



There are  $2^2=4$  possible decision functions  $\delta_{cs}$  for Check Smoke:

•Decision function needs to specify a value for each instantiation of parents

#### CheckSmoke

Report	<i>δ<sub>cs</sub></i> 1	δ <sub>cs</sub> 2	<b>δ</b> <sub>cs</sub> 3	<b>δ</b> <sub>cs</sub> 4
т	Т	Т	F	F
F	Т	F	Т	F

#### **Policies for Sequential Decision Problems**

**Definition (Policy)** A policy  $\pi$  is a sequence of  $\delta_1, \ldots, \delta_n$  decision functions  $\delta_i : \operatorname{dom}(pa(D_i)) \to \operatorname{dom}(D_i)$ 

I.e., when the agent has observed  $o \in \text{dom}(pD_i)$ , it will do  $\delta_i(o)$ 

There are  $2^8$ =256 possible decision functions  $\delta_{cs}$  for Call:

	R=t, CS=t, SS=t	R=t, CS=t, SS=f	R=t, CS=f, SS=t	R=t, CS=f, SS=f	R=f, CS=t, SS=t	R=f, CS=t, SS=f	R=f, CS=f, SS=t	R=f, CS=f, SS=f
$\delta_{call}$ 1(R)	Т	Т	Т	Т	Т	Т	Т	Т
$\delta_{call}$ 2(R)	Т	Т	Т	Т	Т	Т	Т	F
$\delta_{call}$ 3(R)	Т	Т	Т	Т	Т	Т	F	Т
$\delta_{call}$ 4(R)	Т	Т	Т	Т	Т	Т	F	F
$\delta_{call}$ 5(R)	Т	Т	Т	Т	Т	F	Т	Т
$\delta_{call}$ 256(R)	F	F	F	F	F	F	F	F

• If a decision D has k binary parents, how many assignments of values to the parents are there?



- If a decision D has k binary parents, how many assignments of values to the parents are there?
  - 2<sup>k</sup>
- If there are b possible value for a decision variable, how many different decision functions are there for it if it has k binary parents?

$$2^{kp} b^{*}2^{k} b^{2^{k}} 2^{k^{b}}$$

- If a decision D has k binary parents, how many assignments of values to the parents are there?
  - 2<sup>k</sup>
- If there are b possible value for a decision variable, how many different decision functions are there for it if it has k binary parents?
  - b<sup>2<sup>k</sup></sup>, because there are 2<sup>k</sup> possible instantiations for the parents and for every instantiation of those parents, the decision function could pick any of b values
- If there are *d* decision variables, each with *k* binary parents and *b* possible actions, how many policies are there?



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- If there are d decision variables, each with k binary parents and b possible actions, how many policies are there?
  - (b<sup>2<sup>k</sup></sup>)<sup>d</sup>, because there are b<sup>2<sup>k</sup></sup> possible decision functions for each decision, and a policy is a combination of d such decision functions

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#### Expected Utility and Optimality of Policies

- Computing the Optimal Policy by Variable Elimination
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# Possible worlds satisfying a policy

#### **Definition (Satisfaction of a policy)**

A possible world w satisfies a policy  $\pi$ , written w  $\models \pi$ , if the value of each decision variable in w is the value selected by its decision function in policy  $\pi$  (when applied to w)

- Consider our previous example policy:
  - Check smoke (i.e. set CheckSmoke=true) if and only if Report=true
  - Call if and only if Report=true, CheckSmoke=true, SeeSmoke=true
- Does the following possible world satisfy this policy?

   tampering, fire, alarm, leaving, report, smoke, checkSmoke, seeSmoke, call



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- Do the following possible worlds satisfy this policy?
   ¬tampering, fire, alarm, leaving, report, smoke, checkSmoke, seeSmoke, call
  - Yes! Conditions are satisfied for each of the policy's decision functions

-tampering, fire, alarm, leaving, report, smoke, checkSmoke, seeSmoke, -call



# Possible worlds satisfying a policy

#### **Definition (Satisfaction of a policy)**

A possible world w satisfies a policy  $\pi$ , written w  $\models \pi$ , if the value of each decision variable in w is the value selected by its decision function in policy  $\pi$  (when applied to w)

• Consider our previous example policy:

Yes

No

- Check smoke (i.e. set CheckSmoke=true) if and only if Report=true
- Call if and only if Report=true, CheckSmoke=true, SeeSmoke=true
- Do the following possible worlds satisfy this policy?

   tampering, fire, alarm, leaving, report, smoke, checkSmoke, seeSmoke, call
  - Yes! Conditions are satisfied for each of the policy's decision functions

-tampering, fire, alarm, leaving, report, smoke, checkSmoke, seeSmoke, -call

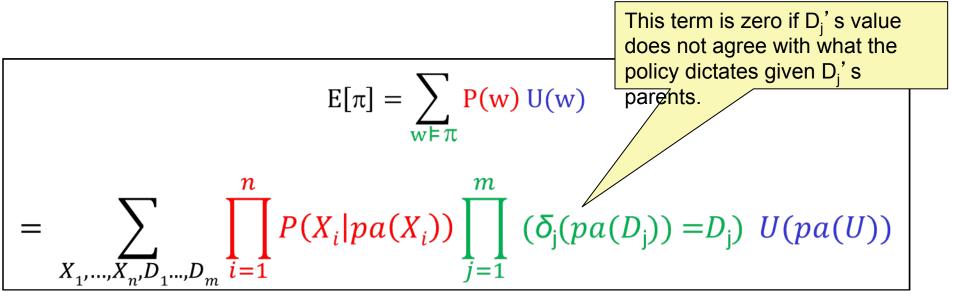
• No! The policy says to call if Report and CheckSmoke and SeeSmoke all true

-tampering,fire,alarm,leaving,-report,-smoke,-checkSmoke,-seeSmoke,-call

• Yes! Policy says to neither check smoke nor call when there is no report

### Expected utility of a policy

Definition (expected utility of a policy) The expected utility  $E[\pi]$  of a policy  $\pi$  is:  $E[\pi] = \sum_{w \models \pi} P(w) U(w)$ 



## Optimality of a policy

Definition (expected utility of a policy) The expected utility  $E[\pi]$  of a policy  $\pi$  is:  $E[\pi] = \sum_{w \models \pi} P(w) U(w)$ 

### **Definition (optimal policy)** An optimal policy $\pi_{max}$ is a policy whose expected utility is maximal among all possible policies $\prod$ : $\pi_{max} \in \operatorname*{argmax}_{\pi \in \prod} E[\pi]$ $\pi \in \prod$

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Computing the Optimal Policy by Variable Elimination

• Summary & Perspectives

One last operation on factors: maxing out a variable

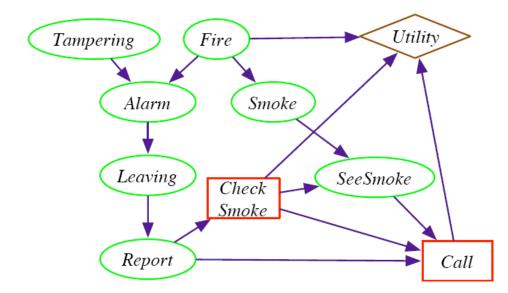
- Maxing out a variable is similar to marginalization
  - But instead of taking the sum of some values, we take the max

One last operation on factors: maxing out a variable

- Maxing out a variable is similar to marginalization
  - But instead of taking the sum of some values, we take the max

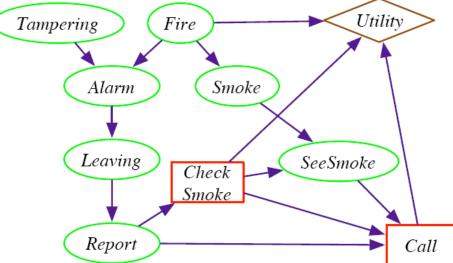
## The no-forgetting property

- A decision network has the no-forgetting property if
  - Decision variables are totally ordered:  $D_1, ..., D_m$
  - If a decision  $D_i$  comes before  $D_j$ , then
    - D<sub>i</sub> is a parent of D<sub>j</sub>
    - any parent of D<sub>i</sub> is a parent of D<sub>i</sub>



## Idea for finding optimal policies with VE

- Idea for finding optimal policies with variable elimination (VE): Dynamic programming: precompute optimal future decisions
  - Consider the last decision D to be made
    - Find optimal decision D=d for each instantiation of D's parents
      - For each instantiation of D's parents, this is just a single-stage decision problem
    - Create a factor of these maximum values: max out D
      - I.e., for each instantiation of the parents, what is the best utility I can achieve by making this last decision optimally?
    - Recurse to find optimal policy for reduced network (now one less decision)

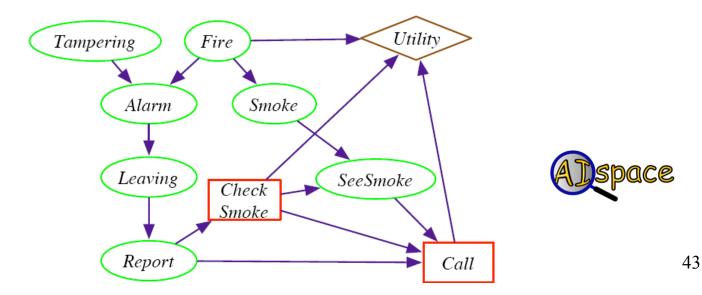


## Finding optimal policies with VE

- 1. Create a factor for each CPT and a factor for the utility
- 2. While there are still decision variables
  - 2a: Sum out random variables that are not parents of a decision node.
    - E.g Tampering, Fire, Alarm, Smoke, Leaving
  - 2b: Max out last decision variable D in the total ordering
    - Keep track of decision function
- 3. Sum out any remaining variable:

this is the expected utility of the optimal policy.

This is Algorithm VE\_DN in P&M, Section 9.3.3, p. 393



# Computational complexity of VE for finding optimal policies

• We saw:

For *d* decision variables (each with *k* binary parents and *b* possible actions), there are  $(b^{2^k})^d$  policies

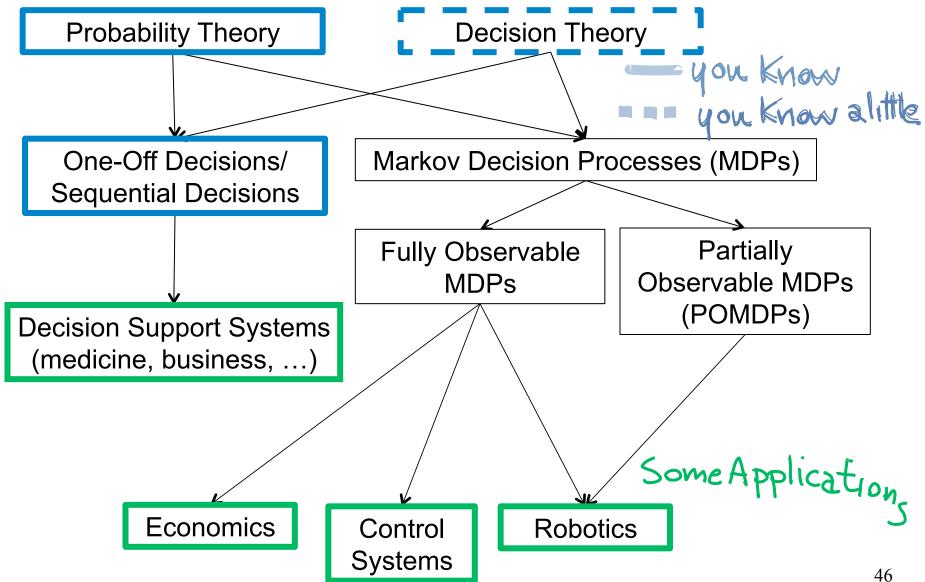
- All combinations of  $(b^{2^k})$  decision functions per decision
- Variable elimination saves the final exponent:
  - Dynamic programming: consider each decision functions only once
  - Resulting complexity:  $O(d * b^{2^k})$
  - Much faster than enumerating policies (or search in policy space), but still doubly exponential
  - CS422: approximation algorithms for finding optimal policies

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Summary & Perspectives

## **Big Picture: Planning under Uncertainty**



## **Decision Theory: Decision Support Systems**

#### E.g., Computational Sustainability

- New interdisciplinary field, AI is a key component
  - Models and methods for decision making concerning the management and allocation of resources
  - to solve most challenging problems related to sustainability
- Often constraint optimization problems. E.g.
  - Energy: when are where to produce green energy most economically?
  - Which parcels of land to purchase to protect endangered species?
  - Urban planning: how to use budget for best development in 30 years?



Source: http://www.computational-sustainability.org/ 47

## **Planning Under Uncertainty**

- Learning and Using POMDP models of Patient-Caregiver Interactions During Activities of Daily Living
- Goal: Help older adults living with cognitive disabilities (such as Alzheimer's) when they:
  - forget the proper sequence of tasks that need to be completed
  - lose track of the steps that they have already completed



Source: Jesse Hoey UofT 2007

## **Planning Under Uncertainty**

#### Helicopter control: MDP, reinforcement learning

(states: all possible positions, orientations, velocities and angular velocities)



Source: Andrew Ng

## **Planning Under Uncertainty**

# Autonomous driving: DARPA Urban Challenge - Stanford's Junior



Source: Sebastian Thrun

## Learning Goals For Today's Class

- Sequential decision networks
  - Represent sequential decision problems as decision networks
  - Explain the non forgetting property
- Policies
  - Verify whether a possible world satisfies a policy
  - Define the expected utility of a policy
  - Compute the number of policies for a decision problem
  - Compute the optimal policy by Variable Elimination