Logic: Bottom-up & Top-down proof procedures

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Lecture Overview

Recap: Soundness, Completeness,
Bottom-up proof procedure

- Bottom-up Proof Procedure
  - Soundness proof
  - Completeness proof

- Top-down Proof Procedure

(Propositional) Logic: Review of Key ideas

- Given a domain that can be represented with n propositions, how many interpretations are there?
  - 2^n interpretations (similar to possible worlds)
- If you do not know anything about the domain you could be in any of those interpretations
- If you know that some logical formulae are true (your KB), you know that you can only be in interpretations in which those formulae hold (i.e. in …………… of KB)

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- If you know that some logical formulae are true (your KB), you know that you can only be in interpretations in which those formulae hold (i.e. in models of KB)
- It is useful to know what else is true in all those models

Definition (logical consequence)
If KB is a set of clauses and g is a conjunction of atoms,
g is a logical consequence of KB, written KB $\models$ g,
if g is true in every model of KB

- Example: KB = $\{h \leftarrow a, a, d \leftarrow c\}$. For which g is KB $\models$ g true?

(Propositional) Logic: Review of Key ideas

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- If you do not know anything about the domain you could be in any of those interpretations
- If you know that some logical formulae are true (your KB), you know that you can only be in interpretations in which those formulae hold (i.e. in models of KB)
- It would be nice to know what else is true in all those models

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- Example: KB = {h $\leftarrow$ a, a, d $\leftarrow$ c}. Then KB $\models$ a and KB $\models$ h.

Intended interpretation

- User chooses task domain: intended interpretation.
  - This is the interpretation of the symbols the user has in mind
- User tells the system clauses (the knowledge base KB)
  - Each clause is true in the user's intended interpretation
  - Thus, the intended interpretation is a model
- The computer does not know the intended interpretation
  - But if it can derive something that's true in all models, then it is true in the intended interpretation
  - Once more, we want to derive logical consequences

Logical consequence

Definition (logical consequence)
If KB is a set of clauses and g is a conjunction of atoms,
g is a logical consequence of KB, written KB ⊧ g,
if g is true in every model of KB

- If KB ⊧ g, then …   (multiple answers correct)

g is true in the intended interpretation

There is at least one model of KB in which g is true

g is true in every model of KB

g is true in some models of KB,
but not necessarily the intended interpretation

Logical consequence

> **Definition (logical consequence)**
> If KB is a set of clauses and g is a conjunction of atoms,
> g is a logical consequence of KB, written KB $\models$ g,
> if g is true in every model of KB

- If KB $\models$ g, then …
  - g is true in every model of KB (by definition)
  - The intended interpretation is one of these models, so g is also true in it
- Notice there is always at least one model of any PDCL theory! Q: What is it?

Logical consequence

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- If KB $\models$ g, then …
  - g is true in every model of KB (by definition)
  - The intended interpretation is one of these models, so g is also true in it
- Notice there is always at least one model of any PDCL theory! Q: What is it?
- A: The interpretation with every atom true is always a model. Why? Because each clause is true in that int'n.

Recap: proofs, soundness, completeness

- A proof is a mechanically derivable demonstration that a formula logically follows from a knowledge base.

Definition (derivability with a proof procedure)
Given a proof procedure P, KB $\vdash_P$ g means g can be derived from knowledge base KB with proof procedure P.

- We want our proof procedures to be sound and complete

Definition (soundness)
A proof procedure P is sound if KB $\vdash_P$ g implies KB $\models$ g.

sound: every atom g that P derives follows logically from KB

Definition (completeness)
A proof procedure P is complete if KB $\models$ g implies KB $\vdash_P$ g.

complete: every atom g that logically follows from KB is derived by P

Example: an unsound proof procedure

- Unsound proof procedure U:
  - U derives every atom in KB: for any g that appears in KB, KB $\vdash_U$ g
- Proof procedure U is unsound:
  - There are atoms it derives that do not logically follow from KB
  - E.g. KB = {a $\leftarrow$ b}.

    It will derive a and b, but neither of them logically follows from KB
  - Thus KB $\vdash_U$ g does not imply KB $\models$ g $\rightarrow$ unsound
- Proof procedure U is complete:
  - It will not miss any atoms since it derives every atom g
  - Thus KB $\models$ g implies KB $\vdash_U$ g $\rightarrow$ complete

Example: an incomplete proof procedure

- Incomplete proof procedure I:
  - I derives nothing: there is no atom g such that KB $\vdash_I$ g

- Proof procedure I is sound:
  - It does not derive any atom at all,
    so every atom it derives follows from KB
  - Thus KB $\vdash_I$ g implies KB $\models$ g $\rightarrow$ sound

- Proof procedure I is incomplete:
  - It will miss atoms that logically follow from KB
  - E.g. KB = {a}: KB $\models$ a, but not KB $\vdash_I$ a
  - Thus KB $\models$ g does not imply KB $\vdash_I$ g $\rightarrow$ incomplete

Recap: Bottom-up proof procedure

KB $\vdash_{BU}$ g if and only if g $\in$ C at the end of the following procedure.

C := {};

repeat

select clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB such that $b_i \in C$ for all i, and $h \notin C$;

C := C $\cup$ {h}

until no more clauses can be selected.

Bottom-up proof procedure: example

C := {};
repeat
    select clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB
        such that $b_i \in C$ for all i, and $h \notin C$;
    $C := C \cup \{h\}$
until no more clauses can be selected. KB $\vdash_{BU}$ g if and only if $g \in C$

$a \leftarrow b \wedge c$
$a \leftarrow e \wedge f$
$b \leftarrow f \wedge k$
$c \leftarrow e$
$d \leftarrow k$
e.
$f \leftarrow j \wedge e$
$f \leftarrow c$
$j \leftarrow c$

{}
{e}
{c,e}
{c,e,f}
{c,e,f,j}
{a,c,e,f,j}

Done.

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Soundness of bottom-up proof procedure BU

Definition (soundness)

A proof procedure P is sound if KB $\vdash_P$ g implies KB $\models$ g.

sound: every atom g that P derives follows logically from KB

C := {};
repeat
select clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB
such that $b_i \in C$ for all i, and $h \notin C$;
C := C $\cup$ {h}
until no more clauses can be selected. KB $\vdash_{BU}$ g if and only if $g \in C$

What do we need to prove to show that BU is sound ?

Soundness of bottom-up proof procedure BU

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until no more clauses can be selected. KB $\vdash_{BU}$ g if and only if $g \in C$

What do we need to prove to show that BU is sound ?

If $g \in C$ at the end of BU procedure,
then g is true in all models of KB (KB $\models$ g)

Soundness of bottom-up proof procedure BU

```
C := {};
repeat
        select clause h ← b1 ∧ … ∧ bm in KB
              such that bi ∈ C for all i, and h ∉ C;
        C := C ∪ {h}
until no more clauses can be selected.
```

For soundness of bottom-up proof procedure BU: prove

If $g \in C$ at the end of BU procedure,
then g is true in all models of KB (KB $\models$ g)

By contradiction: Suppose there is a g such that KB $\vdash$ g but not KB $\models$ g.

- Let h be first atom added to C that's not true in every model of KB
  - In particular, suppose I is a model of KB in which h isn't true.
- There must be a clause in KB of form $h \leftarrow b_1 \wedge \ldots \wedge b_m$
- Each b_i is true in I. h is false in I. So this clause is false in I.
- Thus, I is not a model of KB. Contradiction: thus no such g exists

Soundness of bottom-up proof procedure BU

- The previous proof of soundness of BU proof procedure was by contradiction.
- Now we will give a direct proof by induction….

Soundness of bottom-up proof procedure BU

C := {};
repeat
    select clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB
        such that $b_i \in C$ for all i, and $h \notin C$;
    $C := C \cup \{h\}$
until no more clauses can be selected. KB $\vdash_{BU}$ g if and only if $g \in C$

Inductive proof using inductive hypothesis IH:

IH: if $g \in C$ at loop iteration n, **then** g is true in all models of KB (KB $\models$ g)

Base case: "IH holds for n=0". C = {}, so IH holds trivially

Inductive case: "if IH holds for n, it holds for n+1".

- Here: "if IH held before a loop iteration, it holds afterwards"
- The only new element in C is h, so we only need to prove KB $\models$ h
- $b_1, \ldots, b_m$ were in C before, so by IH we know KB $\models b_1 \wedge \ldots \wedge b_m$
- In every model, "$b_1 \wedge \ldots \wedge b_m$" is true and "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" is true
    - Thus, in every model, h is true. Done. KB $\vdash_{BU}$ g implies KB $\models$ g

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Minimal Model

C := {};
repeat
  select clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB
    such that $b_i \in C$ for all i, and $h \notin C$;
  $C := C \cup \{h\}$
until no more clauses can be selected. KB $\vdash_{BU}$ g if and only if $g \in C$

The C at the end of BU procedure is a fixed point:

- Further applications of our rule of derivation will not change C!

Definition (minimal model)
The minimal model MM is the interpretation in which every element of BU's fixed point C is true and every other atom is false.

Lemma: minimal model MM is a model of KB

Definition (minimal model)
The minimal model MM is the interpretation in which every element of BU's fixed point C is true and every other atom is false.

Definition (model)
A **model** of a knowledge base KB is an interpretation in which every clause in KB is true.

Proof by contradiction.

Assume (for contradiction) that MM is not a model of KB.

- Then there must exist some clause in KB which is false in MM
  - Like every clause in KB, it is of the form $h \leftarrow b_1 \wedge \ldots \wedge b_m$ (with $m \geq 0$).
- $h \leftarrow b_1 \wedge \ldots \wedge b_m$ can only be false in MM if each b_i is true in MM and h is false in MM.
  - Since each b_i is true in MM, each b_i must be in C as well.
  - BU would add h to C, so h would be true in MM
  - Contradiction! Thus, MM is a model of KB

Completeness of bottom-up procedure

Definition (completeness)

A proof procedure P is complete if KB $\models$ g implies KB $\vdash_P$ g.

complete: everything that logically follows from KB is derived

What do we need to prove to show that BU is complete?

Completeness of bottom-up procedure

Definition (completeness)

A proof procedure P is complete if $KB \models g$ implies $KB \vdash_P g$.

complete: everything that logically follows from KB is derived

What do we need to prove to show that BU is complete?

If g is true in all models of KB ($KB \models g$)
then $g \in C$ at the end of BU procedure ($KB \vdash_{BU} g$)

Direct proof based on Lemma about minimal model:

- Suppose $KB \models g$. Then g is true in all models of KB.
- Thus g is true in the minimal model.
- Thus $g \in C$ at the end of BU procedure.
- Thus $KB \vdash_{BU} g$. Done. $KB \models g$ implies $KB \vdash_{BU} g$

Summary for bottom-up proof procedure BU

- BU is sound:
  it derives only atoms that logically follow from KB

- BU is complete:
  it derives all atoms that logically follow from KB

- Together:
  it derives exactly the atoms that logically follow from KB

- And, it is efficient!
  - Linear in the number of clauses in KB
    - Each clause is used maximally once by BU

Learning Goals Up To Here

- PDCL syntax & semantics
  - Verify whether a logical statement belongs to the language of propositional definite clauses
  - Verify whether an interpretation is a model of a PDCL KB.
  - Verify when a conjunction of atoms is a logical consequence of a knowledge base
- Bottom-up proof procedure
  - Define/read/write/trace/debug the Bottom Up (**BU**) proof procedure
  - Prove that the BU proof procedure is sound and complete

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Bottom-up vs. Top-down

Bottom-up

KB $\Rightarrow$ C

g is proved if $g \in C$

When does BU look at the query g?

In every loop iteration

Never

At the end

At the beginning

Bottom-up vs. Top-down

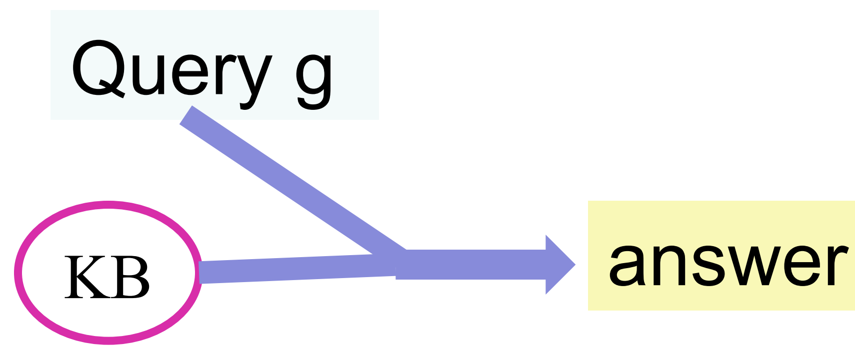
- **Key Idea of top-down:** search backward from a query g to determine if it can be derived from *KB*.

Bottom-up

g is proved if $g \in C$

When does BU look at the query g?
- Never
- It derives the same C regardless of the query

Top-down



We'll see how g is proved

TD performs a backward search starting at g

Top-down Ground Proof Procedure

Idea: search backward from a query to determine if it is a logical consequence of KB

An answer clause is of the form: $yes \leftarrow a_1 \wedge \ldots \wedge a_m$ where $a_1, \ldots, a_m$ are atoms

We express the query as an answer clause

– E.g. query $q_1 \wedge \ldots \wedge q_k$ is expressed as $yes \leftarrow q_1 \wedge \ldots \wedge q_k$

Basic operation: SLD Resolution of an answer clause

$$yes \leftarrow c_1 \wedge \ldots \wedge c_{i-1} \wedge c_i \wedge c_{i+1} \ldots \wedge c_m$$

on an atom c_i with another clause

$$c_i \leftarrow b_1 \wedge \ldots \wedge b_p$$

yields the clause

$$yes \leftarrow c_1 \wedge \ldots \wedge c_{i-1} \wedge b_1 \wedge \ldots \wedge b_p \wedge c_{i+1} \ldots \wedge c_m$$

Rules of derivation in top-down and bottom-up

Top-down:

SLD Resolution

$$\frac{yes \leftarrow c_1 \wedge \ldots \wedge c_m \qquad\qquad c_i \leftarrow b_1 \wedge \ldots \wedge b_p}{yes \leftarrow c_1 \wedge \ldots \wedge c_{i-1} \wedge b_1 \wedge \ldots \wedge b_p \wedge c_{i+1} \ldots \wedge c_m}$$

Bottom-up:

Generalized modus ponens

$$\frac{h \leftarrow b_1 \wedge \ \ldots \wedge b_m \qquad\qquad b_1 \wedge \ \ldots \wedge b_m}{h}$$

SLD Derivations

- An answer is an answer clause with m = 0.

  yes $\leftarrow$.

- A successful derivation from KB of query $?q_1 \wedge \dots \wedge q_k$ is a sequence of answer clauses $\gamma_0, \gamma_1, \dots, \gamma_n$ such that
  - γ_0 is the answer clause yes $\leftarrow q_1 \wedge \dots \wedge q_k$.
  - γ_i is obtained by resolving γ_{i-1} with a clause in KB, and
  - γ_n is an answer yes $\leftarrow$

- An unsuccessful derivation from KB of query $?q_1 \wedge \dots \wedge q_k$
  - We get to something like yes $\leftarrow b_1 \wedge \dots \wedge b_k$.
  - There is no clause in KB with any of the b_i as its head

Top-down Proof Procedure for PDCL

To solve the query $? \; q_1 \wedge \ldots \wedge q_k$:

ac:= yes $\leftarrow$ body, where body is $q_1 \wedge \ldots \wedge q_k$

repeat

select $q_i \in$ body;

choose clause $C \in$ KB, C is $q_i \leftarrow b_c$;

replace q_i in body by b_c

until ac is an answer (fail if no clause with q_i as head)

select: any choice will work

("don't care" non-determinism)

choose: non-deterministic, have to pick the right one

("don't know" non-determinism)

Example: successful derivation

$a \leftarrow b \wedge c.$ 1 $\boxed{a \leftarrow e \wedge f.}$ $b \leftarrow f \wedge k.$

4 $\boxed{c \leftarrow e.}$ $d \leftarrow k$ 3 5 $\boxed{e.}$

$f \leftarrow j \wedge e.$ 2 $\boxed{f \leftarrow c.}$ $j \leftarrow c.$

Query: ?a

γ_0: yes $\leftarrow$ a

γ_1: yes $\leftarrow$ e $\wedge$ f

γ_2: yes $\leftarrow$ e $\wedge$ c

γ_3: yes $\leftarrow$ c

γ_4: yes $\leftarrow$ e

γ_5: yes $\leftarrow$

Example: failing derivation

1 $\boxed{a \leftarrow b \wedge c.}$ $a \leftarrow e \wedge f.$ 2 $\boxed{b \leftarrow f \wedge k.}$

4,6 $\boxed{c \leftarrow e.}$ $d \leftarrow k$ 5, 7 $\boxed{e.}$

$f \leftarrow j \wedge e.$ 3 $\boxed{f \leftarrow c.}$ $j \leftarrow c.$

Query: ?a

γ_0: yes $\leftarrow$ a

γ_1: yes $\leftarrow$ b $\wedge$ c

γ_2: yes $\leftarrow$ f $\wedge$ k $\wedge$ c

γ_3: yes $\leftarrow$ c $\wedge$ k $\wedge$ c

γ_4: yes $\leftarrow$ e $\wedge$ k $\wedge$ c

γ_5: yes $\leftarrow$ k $\wedge$ c

γ_6: yes $\leftarrow$ k $\wedge$ e

γ_7: yes $\leftarrow$ k

There is no rule with k as its head, thus … fail

Correspondence between BU and TD proofs

If the following is a top-down derivation in a given KB, what would be the bottom-up derivation of the same query?

yes ← a.	{}
yes ← b ∧ f.	{h}
yes ← b ∧ g ∧ h.	{g,h}
yes ← c ∧ d ∧ g ∧ h.	{d,g,h}
yes ← d ∧ g ∧ h.	{c,d,g,h}
yes ← g ∧ h.	{b,c,d,g,h}
yes ← h.	{b,c,d,f,g,h}
yes ← .	{a,b,c,d,f,g,h}

Is the Top-down procedure sound and complete?

- Yes, since there is a 1:1 correspondence between top-down and bottom-up proofs.
  - The two methods derive exactly the same atoms (if the SLD resolution picks the successful derivations)
  - And the bottom-up procedure is sound and complete
  - Therefore the top-down procedure is sound and complete