Menu January 12, 2016

Topics:
- Linear Filters
- Correlation/Convolution
- Filter Examples: Box, Gaussian

Reading:
- Today: Forsyth & Ponce (2nd ed.) 4.1, 4.5

Handouts:
- Assignment 2: Gaussian Filters due Tuesday, January 19.

Reminders:
- You should have completed Assignment 1 by now.
- piazza: https://piazza.com/ubc.ca/winterterm22015/cpsc425/
Today’s “Fun” Example: Automimicry
The Peablue (*Lampides boeticus*) is a small butterfly. Notice the eyespot and pseudo-antennae on the hindwing.

---

Image credit:
Lecture 2: Re-cap

- We take a “physics-based” approach to image formation — Treat camera as an instrument that takes measurements of the 3D world

- Basic abstraction is the pinhole camera

- Lenses overcome limitations of the pinhole model while trying to preserve it as a useful abstraction

- When maximum accuracy required, it is necessary to model additional details of each particular camera (and camera setting) — Aside: This is called camera calibration
Lecture 2 Re-cap (cont’d): Projection Equations

3D world point, $P[x, y, z]$, projects to 2D image point $P'[x', y']$ where

**Perspective**

$x' = f' \frac{x}{z}$

$y' = f' \frac{y}{z}$

**Weak Perspective**

$x' = m x$

$y' = m y$

$m = \frac{f'}{z_0}$

**Orthographic**

$x' = x$

$y' = y$
Lecture 2 Re-cap (cont’d): Snell’s Law

\[ n_1 \sin \alpha_1 = n_2 \sin \alpha_2 \]
Thin lens equation

\[ \frac{1}{z'} - \frac{1}{z} = \frac{1}{f} \]

does not characterize the relationship between \( f, z \) and \( z' \)

Some “aberrations and distortions” persist. For example:
— index of refraction depends on wavelength, \( \lambda \), of light
— vignetting reduces image brightness (gradually) away from the image center

The human eye functions much like a camera
Framework for Today’s Topic

Problem:
Digital image processing: take a digital image as input and compute a modified image as output

Key Idea(s): Consider computations that are both linear and shift invariant

Alternatives:
— (eventually) include non linear computations

Theory:
Linear, shift invariant systems

Practical Detail(s): Implement correlation efficiently

“Gotchas:”
— difference between correlation and convolution
— boundary effects
Linear Filters

Let \( I(X, Y) \) be an \( n \times n \) digital image

Let \( F(X, Y) \) be another \( m \times m \) digital image (our “filter”)

For convenience, assume \( m \) is odd. (Here, \( m = 5 \))
Linear Filters (cont’d)

Let \( k = \left\lfloor \frac{m}{2} \right\rfloor \)

Compute a new image, \( l'(X, Y) \), as follows:

\[
l'(X, Y) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} F(i, j) \cdot I(X + i, Y + j)
\]
For a given $X$ and $Y$, superimpose the filter on the image centered at $(X, Y)$. 
Compute the new pixel value, $I'(X, Y)$, as the sum of $m \times m$ values, where each value is the product of the original pixel value in $I(X, Y)$ and the corresponding value in the filter.
Consider another $X$ and $Y$. Again superimpose the filter on the image centered at $(X, Y)$

The computation is repeated for each $(X, Y)$
Example
Linear Filters (cont’d)

\[
I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X + i, Y + j)
\]

For each \(X\) and \(Y\), superimpose the filter, \(F(X, Y)\), on the image centered at \((X, Y)\)

Compute the new pixel value, \(I'(X, Y)\), as the sum of \(m \times m\) values, where each value is the product of the original pixel value in \(I(X, Y)\) and the corresponding value in the filter
Linear Filters (cont’d)

Let’s also do the accounting…

\[ l'(X, Y) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} F(i, j) \cdot l(X + i, Y + j) \]

At each pixel, \((X, Y)\), there are \(m \times m\) multiplications

There are \(n \times n\) pixels, \((X, Y)\)

Total \(m^2 \times n^2\) multiplications

When \(m\) is a fixed, small constant, this is \(O(n^2)\). But, when \(m \approx n\), this is \(O(n^4)\)
Linear Filters: Boundary Effects
Linear Filters: Boundary Effects

3 standard ways to deal with boundaries

1. Ignore these locations
   — Make the computation undefined for the top and bottom $k$ rows and the leftmost and rightmost $k$ columns

2. Pad the image with zeros
   — Return zero whenever a value of $I$ is required at some position outside the defined limits of $X$ and $Y$

3. Assume periodicity
   — The top row wraps around to the bottom row; the leftmost column wraps around to the rightmost column
A short exercise
Example 1: Warm Up

Original

Filter

Result
Example 1: Warm Up (cont’d)

Original

Filter

Result (no change)
Example 2:
Example 2 (cont’d):

<table>
<thead>
<tr>
<th>Original</th>
<th>Filter</th>
<th>Result (shift left by 1 pixel)</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1.png" alt="Original Image" /></td>
<td><img src="image2.png" alt="Filter" /></td>
<td><img src="image3.png" alt="Result" /></td>
</tr>
</tbody>
</table>
Example 3:

Original

Filter

Result

\[
\begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

(filter sums to 1)
Example 3 (cont’d):

Original  Filter  Result

(blur with a box filter)
Example 4:

Original

Filter

Result

(filter sums to 1)
Example 4 (cont’d):

Original  
Filter  
Result  
(sharpening)
Example 4 (cont’d):

Sharpening

Before

After
Linear Filters (cont’d)

Definition: Correlation

\[ I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X + i, Y + j) \]

Definition: Convolution

\[ I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X - i, Y - j) \]

\[ = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(-i, -j) I(X + i, Y + j) \]

Note: If \( F(X, Y) = F(-X, -Y) \) then correlation \( \equiv \) convolution
Linear Filters (cont’d)

Definition: Correlation

\[ I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X + i, Y + j) \]

Definition: Convolution

\[ I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X - i, Y - j) \]

\[ = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(-i, -j) I(X + i, Y + j) \]

Note: If \( F(X, Y) = F(-X, -Y) \) then correlation \( \equiv \) convolution
Linear Filters: Properties

Let $\otimes$ denote convolution. Let $I(X, Y)$ be a digital image

**Superposition:**

Let $F_1$ and $F_2$ be digital filters

\[
(F_1 + F_2) \otimes I(X, Y) = F_1 \otimes I(X, Y) + F_2 \otimes I(X, Y)
\]

**Scaling:**

Let $F$ be a digital filter and let $k$ be a scalar

\[
(kF) \otimes I(X, Y) = F \otimes (kI(X, Y)) = k(F \otimes I(X, Y))
\]

**Shift Invariant:**

Output is local (i.e., no dependence on absolute position)

Forsyth & Ponce (1st and 2nd ed.) say,

"the response to a translated stimulus is just a translation of the response to the stimulus"

An operation is linear if satisfies both superposition and scaling
Linear Filters: Shift invariant

Output does not depend on absolute position
Linear Systems: Characterization Theorem

Any linear, shift invariant operation can be expressed as a convolution
Example 5: Smoothing with a Box Filter

- Filter has equal, positive entries that sum to 1

- Replace each pixel with the average of itself and its local neighbourhood
  — Box filter also referred to as average filter
Example 5 (cont’d): Smoothing with a Box Filter

Forsyth & Ponce (2nd ed.) Figure 4.1 (left & middle)
Smoothing with a Box Filter

What happens as we increase the width of the box filter?
Smoothing with a Box Filter

Image credit: Gonzalez and Woods (3rd ed.) Fig. 3.33
Example 6: Smoothing with a Gaussian

- Smoothing with a box doesn’t model lens defocus well
  — Smoothing with a box filter depends on direction
  — Thought experiment: image in which the center point is 1 and every other point is 0

- Smoothing with a (circular) pillbox is a better model for defocus (in geometric optics)

- The Gaussian is a good general smoothing model
  — for phenomena (that are the sum of other small effects)
  — whenever the Central Limit Theorem applies
Example 6 (cont’d): Smoothing with a Gaussian

Idea: Weigh contributions of neighbouring pixels by nearness

2D Gaussian (continuous case):

\[ G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \]

Forsyth & Ponce (2nd ed.)
Figure 4.2
Example 6 (cont’d): Smoothing with a Gaussian

Forsyth & Ponce (2nd ed.) Figure 4.1 (left & right)
Summary

- The correlation of $F(X, Y)$ and $I(X, Y)$ is

$$I'(X, Y) = \sum_{j=-k}^{k} \sum_{i=-k}^{k} F(i, j) I(X + i, Y + j)$$

- Visual interpretation: Superimpose the filter $F$ on the image $I$ at $(X,Y)$, perform an element-wise multiply, and sum up the values.

- Convolution is like correlation except filter “flipped” — When $F(-i, -j) = F(i, j)$ the two are equivalent.

- Characterization Theorem: Any linear, spatially invariant operation can be expressed as a convolution.
Reminders:

Assignment 2 due start of lecture on Tuesday, January 19

piazza: https://piazza.com/ubc.ca/winterterm22015/cpsc425/