Octaves and inharmonicity in piano tuning

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Abstract

I show that equal beating 4:2/6:3 octaves (called 4:2+ octaves) results in an approximately equal beating 2:1 octave as well.

1 Partials, beats, and cents

Let's start by reviewing some basic formulae and definitions.

Suppose we have two single frequencies (sine waves) x and y, measured in Hertz (or any other units). The cent distance c between y and x is per definition

$$c = \frac{1200}{\log 2} \log(y/x), \tag{1}$$

with log the natural logarithm. If x and y are very close we will hear a beat with frequency (beatrate) $b = |y - x|$. The beatrate is given very accurately in terms of the cent difference by

$$b = \frac{\log 2}{1200} cx. \tag{2}$$

For example is $x = 440Hz$ and $c = 1$ we get $b \approx 1/4$, i.e., one beat every 4 seconds for a 1 cent detuned $A4$ unison.

A real piano tone is composed of not just a single frequency but a set of partials. In the absense of inharmonicity the partial frequencies are $x, 2x, 3x, \ldots$ with x the fundamental frequency. In terms of cents the distance of the i-th partial from the fundamental is given by

$$c(i) = \frac{1200}{\log 2} \log(i), \quad i = 1, 2, 3, \ldots \tag{3}$$

In a real piano the partials are all shifted up by a small amount, and effect called inharmonicity. We consider here the Young model [2] and a more realistic modification thereof based on piano data by Robert Scott as implemented in Tunelab [1].

According to this model each partial i is offset by an amount $B(a_i - 1)$ where B is the inharmonicity constant of the note and the coefficients a_i are given by

$$a_i = i^2 \tag{4}$$

in the Young model and are given by

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 4 \\ a_3 &= 8.45 \\ a_4 &= 13.18 \\ a_5 &= 19.72 \\ a_6 &= 27.27 \\ a_7 &= 35.53 \\ a_8 &= 46.25 \end{aligned} \tag{5}$$

by the Tunelab model. Values for higher partials can be found in the Tunelab manual. Formula (3) is thus modified to

$$c(i) = \frac{1200}{\log 2} \log(i) + B(a_i - 1). \tag{6}$$

2 Tuning the octave by partial matching

Consider tuning a note N_2 an octave above a note N_1. Both notes have a partial structure given by (6), but will have different inharmonicity constants B which we call B_1 and B_2. Above the break B_2 will be about twice the value of B_1.

Let's measure all the partials in cents relative to the fundamental of N_1. Also, let's assume we first tune N_2 so that its fundamental is exactly 1200 cents above the fundamental of N_1 and then adding a stretch of s_2 cents, where the value of s_2 is to be determined later. The cent values of the i-th partials of both notes (w.r.t the fundamental of N_1) are now given by

$$\begin{aligned} c_1(i) &= \frac{1200}{\log 2} \log(i) + B_1(a_i - 1) \\ c_2(i) &= 1200 + \frac{1200}{\log 2} \log(i) + B_2(a_i - 1) + s_2. \end{aligned} \tag{7}$$

As a warm-up excercise, let's compute the stretch s_2 first for a 2:1 octave. We want the second partial of N_1 to match the first partial of N_2, so the equation to solve for s_2 is $c_1(2) = c_2(1)$ which reads

$$1200 + B_1(a_2 - 1) = 1200 + s_2, \tag{8}$$

where we have used $a_1 = 1$. We thus get for a 2:1 octave the stretch

$$s_2[2:1] = B_1(a_2 - 1). \tag{9}$$

For example if $N_1 = C4$ and using a typical value $B_1 = 0.404$ we get a stretch of 1.21 cents. We can calculate the theoretical stretch for the 4:2 and 6:3 octaves in the same manner by solving $c_1(4) = c_2(2)$ and $c_1(6) = c_2(3)$ which gives

$$s_2[4:2] = B_1(a_4 - 1) - 3B_2 \tag{10}$$

and

$$s_2[6:3] = B_1(a_6 - 1) - B_2(a_3 - 1). \tag{11}$$

We see that the larger B_2 is, i.e., the steeper the inharmonicity curve is, the less stretch is required for these octaves. Taking the "average" values for C4 and C5 from the Tunelab sample data, $B_1 = 0.404$ and $B_2 = 1.116$ this gives stretches of $s_2[4:2] = 1.57$ and $s_2[6:3] = 2.3$ cents.

3 Tuning the octave by equal beating 4:2 and 6:3

Next we will compute the stretch required for the 4:2+ octave, which has equal beating 4:2 and 6:3, with 4:2 being wide, and 6:3 narrow. First we need the cent difference $\Delta(i)$ between partial i of N_2 and partial $2i$ of note N_1. Using (7) we get

$$\Delta(i) = c_2(i) - c_1(2i) = B_2(a_i - 1) - B_1(a_{2i} - 1) + s_2. \tag{12}$$

A positive $\Delta(i)$ means the $2i : i$ octave is wide, a negative means it is narrow. The beat rate $b(i)$ of this difference according to (2) is given by

$$b(i) = \frac{\log 2}{1200}\Delta(i)2if_0 \tag{13}$$

where f_0 is the frequency of the fundamental of N_1. Note that the frequency of partial $2i$ is not exactly $2if_0$ but the difference is very small and lies beyond the threshold of hum beat speed discrimination. The beat frequency (or rate) of the $2i : i$ octave as given by (13) is positive if the octave is wide, and negative if narrow. We now want to find s_2 such that $b(2) = -b(3)$, i.e., the 4:2 beats wide at the same speed the 6:3 beat narrow. Using (13) and (12) we obtain the stretch s_2

$$
\begin{aligned}
& b(2) + b(3) = 0 \;\Rightarrow \\
& 2\Delta(2) + 3\Delta(3) = 0 \;\Rightarrow \\
& 2B_2(a_2 - 1) - 2B_1(a_4 - 1) + 2s_2 + 3B_2(a_3 - 1) - 3B_1(a_6 - 1) + 3s_2 = 0 \;\Rightarrow \\
& s_2[4:2+] = (B_1(2a_4 + 3a_6 - 5) - B_2(2a_2 + 3a_3 - 5))/5. \qquad (14)
\end{aligned}
$$

If we use the numbers as after (11) we obtain a stretch of 2 cents. However if we used Young's model the stretch would come out to be about 4.2 cents, which is more than twice as much!

With the 4:2+ stretch (14) in had we can now compute the beat rates of the various octave partial matches by using (12) and (13). After some algebra we get the 4:2+ octave 2i:i beat rates

$$b(i) = \frac{f_0 \log 2}{3000}[B_1(2a_4 + 3a_6 - 5a_{2i}) - B_2(2a_2 + 3a_3 - 5a_i)]i. \tag{15}$$

For the Tunelab model this comes out to be for 4:2 as

$$b(2) = b_{4:2} = \frac{f_0 \log 2}{1200}(33.82B_1 - 10.7B_2). \tag{16}$$

Of course be have $b(2) = -b(3)$ which is easy to check. The big question is now, what about 2:1? So let's compute the beat rate difference between 4:2 and 2:1, i.e., $b(2) - b(1)$ using (15). We get

$$b(2) - b(1) = \frac{f_0 \log 2}{3000}[B_1(2a_4 + 3a_6 + 20 - 10a_4) - B_2(2a_2 + 3a_3 - 35)]. \tag{17}$$

Remarkably, for Young's model $a_i = i^2$ the coefficients of B_1 and B_2 are both zero so we have 2:1 beating at exactly the same speed as 4:2 and 6:3. For the tunelab model, plugging in in the table values we obtain instead

$$b(2) - b(1) = \frac{f_0 \log 2}{3000}[1.65B_2 - 3.63B_1]. \tag{18}$$

So if $B_2 = 2.2B_1$ we have 2:1 equal beating, and this relation is almost satisfied on most piano scales. Even if it were not and we had $B_2 = B_1$ or $B_2 = 4B_1$ which are extreme values probably not found on any real piano, we still have a beat speed difference of only 0.1 and 0.14 Hz (or beats per second if you like): virtually equal beating! For comparison the 4:2 (and 6:3) beat speed in this example is 0.5 Hz. On the other hand, the 8:4 octave is quite narrow and beats at 6.4 Hz.

References

[1] Robert Scott. Tunelab piano tuning software. *http://www.tunelab-world.com*.

[2] Robert W. Young. Inharmonicity of Plain Wire Piano Strings. *The Journal of the Acoustical Society of America*, 24(3), 1952.