Efficient algorithms with restricted workspace: shortest paths in grid graphs, using budgeted recursion$^\diamondsuit$

David Kirkpatrick

Department of Computer Science
University of British Columbia

PIMS Workshop on:
Algorithmic Theory of Networks
March, 27-29, 2015

$\diamondsuit$ based on joint work with *Tetsuo Asano*

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Outline

Introduction
- algorithms for shortest (min-weight) paths
- memory-constrained algorithms

Min-weight paths in grid graphs – Asano&Doerr(2011)
- overview of basic algorithm
- applying a good idea recursively

Min-weight paths in grid graphs – Refinements & Extensions
- a different recursive formulation
- budgeted recursion – exploiting a universal sequence
- combining the ideas

Beyond grid graphs...
- min-weight paths in implicit graphs
- min-weight paths in general planar graphs

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Single-source shortest (minimum weight) paths and space-bounded computation

The *topic* lies at the confluence of two fundamental streams in the modern theory of algorithms

- algorithms for minimum-weight paths in graphs
- determining the limits of space-bounded computation, including time-space tradeoffs

Our *model* assumes an input graph provided in read-only memory. Space measures the number of (bounded-capacity) reusable words of working memory. We will write $\tilde{O}(s(n))$ space to acknowledge the fact that words typically have capacity $\Theta(\lg n)$.

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Algorithms for min-weight paths

Finding min-weight paths (in directed graphs with n vertices and m edges)

- general edge weights $O(nm)$ [Bellman-Ford 1950's]
- non-negative edge weights $O(m + n \lg n)$ [Dijkstra, with Fibonacci heaps 1959; 1984]
- *planar* graphs (with non-negative weights) $O(n)$ [Henzinger et al. 1997]
- small integer weights...

All of these are naturally implemented with $\Omega(n + m)$ workspace.

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Memory-constrained algorithms

In addition to the obvious practical advantages of space-efficient algorithms for min-weight paths, the basic graph reachability problem is a core problem in computational complexity theory.

- it is a canonical complete problem for non-deterministic log-space
- the open question **L**=**NL**?, asks if it can be solved deterministically in log-space
- Savitch's algorithm (1970) solves the problem in $O(\lg n)^2)$ space, but requires $n^{\Theta(\lg n)}$ time
- *undirected* graph reachability has a $O(\lg n)$-space solution [Reingold 2008]

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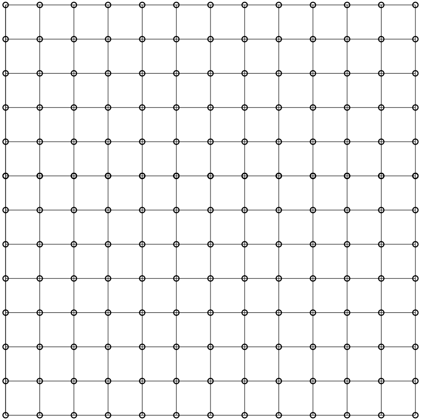
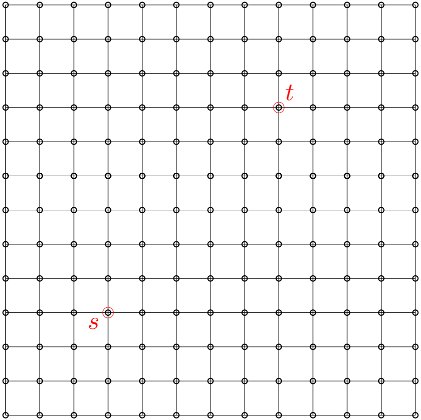
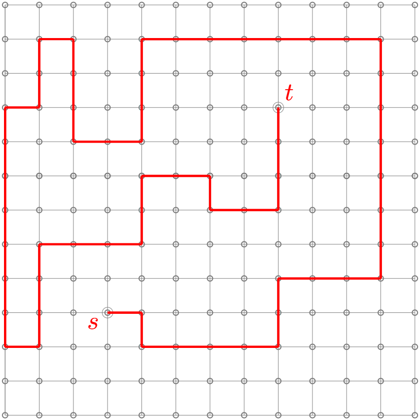
Suppose we are given an edge-weighted grid graph...



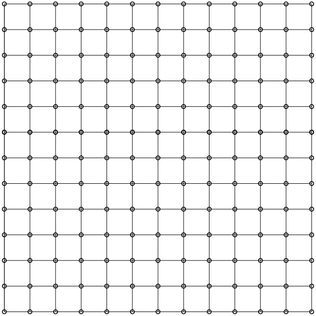
...with two distinguished vertices s and t



We want to find an s-t path of minimum weight

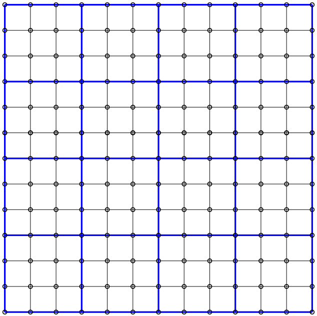


Asano-Doerr algorithm



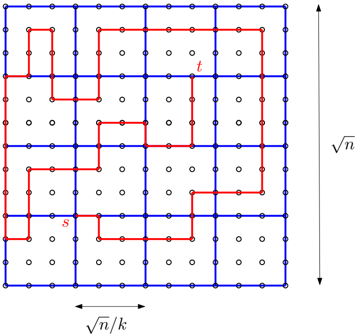
Start with a $\sqrt{n} \times \sqrt{n}$ grid...

Asano-Doerr algorithm (following Fredrickson '87)



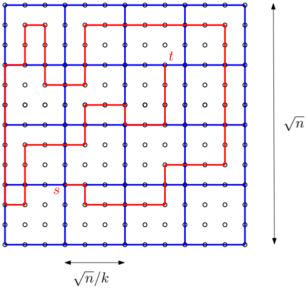
...and partition it into k^2 cells, each of size $\sqrt{n}/k \times \sqrt{n}/k$.

Asano-Doerr algorithm



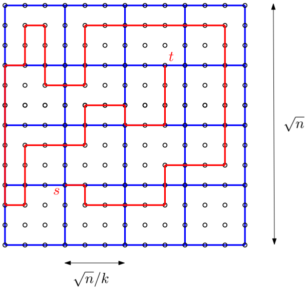
View an *s*-*t* path as a sequence of hops between (cell) boundaries

Asano-Doerr algorithm



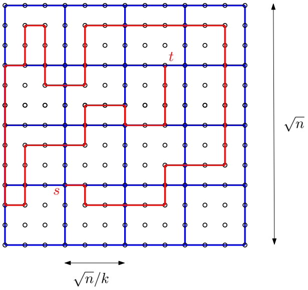
- cell interiors act as quasi-edges connecting boundary vertices
- solve a min-weight path problem on boundary vertices, each "step" of which involves a min-weight path problem (within a cell)

Asano-Doerr algorithm – general edge weights



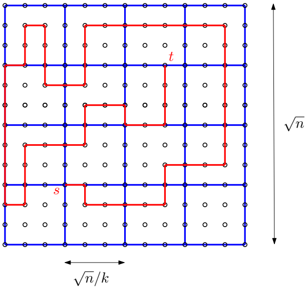
- $O(\sqrt{n}k)$ phases
- each phase involves a "relaxation" of all k^2 quasi-edges
- since each "relaxation" has cost $[(\sqrt{n}/k)^2]^2$, total cost is $O(n^{2.5}/k)$

Asano-Doerr algorithm – non-negative edge weights



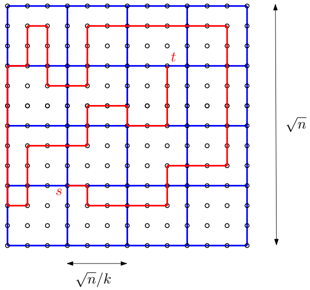
- $O(\sqrt{n}k)$ phases
- each phase involves a "relaxation" of $O(1)$ quasi-edges
- since each "relaxation" costs time $\tilde{O}((\sqrt{n}/k)^2)$, total time is reduced to $\tilde{O}(n^{1.5}/k)$

Asano-Doerr algorithm



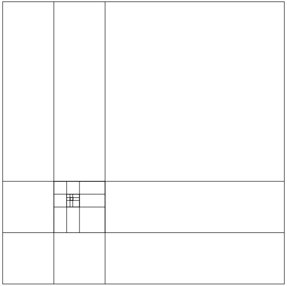
In both cases, space cost is $O(k\sqrt{n} + (\sqrt{n}/k)^2)$,
which is minimized at $O(n^{2/3})$, when $k = n^{1/6}$.

Asano-Doerr algorithm



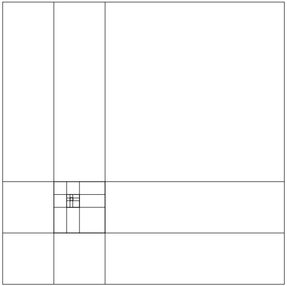
In both cases, space cost is $O(k\sqrt{n} + (\sqrt{n}/k)^2)$, which is minimized at $O(n^{2/3})$, when $k = n^{1/6}$.

Asano-Doerr algorithm – applied recursively



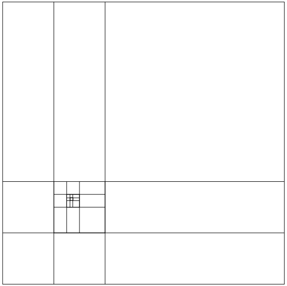
- If the same idea is applied recursively on the cells ($\sqrt{n}/k \times \sqrt{n}/k$ subgrids), with the same splitting factor at m levels of recursion, we get a total time cost of $O(\frac{(\sqrt{n})^m}{k^{m(m+1)/2}} n^2)$ (for general edge weights).

Asano-Doerr algorithm – applied recursively



- The space cost is $O(\sqrt{n}k + n/k^{2m})$, which is minimized when $k = n^{\frac{1}{2(2m+1)}}$, giving space $n^{1/2+\epsilon}$ and time $n^{O(1/\epsilon)}$, when $m = \Theta(1/\epsilon)$.

Asano-Doerr algorithm – optimized



- In fact, if the same idea is applied recursively on the cells with a differentiated splitting factor (chosen to balance the space cost) at each of the m levels of recursion, we get a space cost of $n^{1/2+\epsilon}$ and time $n^{O(\lg(1/\epsilon))}$, when $m = \Theta(\lg(1/\epsilon))$.

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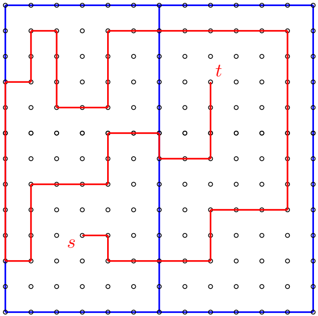
Alternative recursive algorithm



View the *fundamental problem* as one of updating path estimates on boundary vertices (using paths that lie strictly interior to cells).

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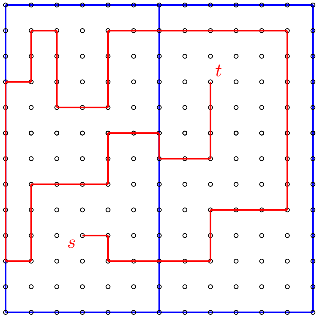

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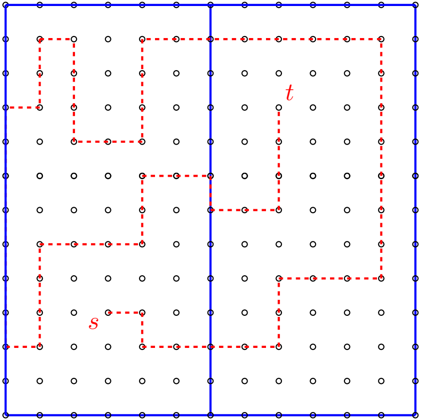

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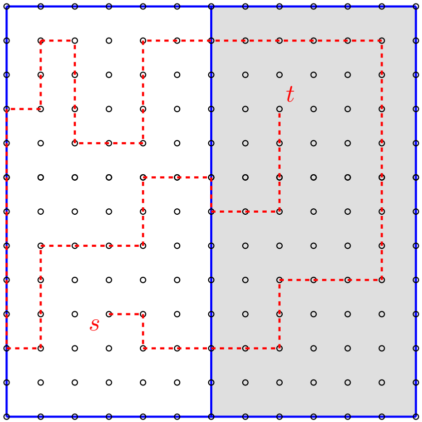

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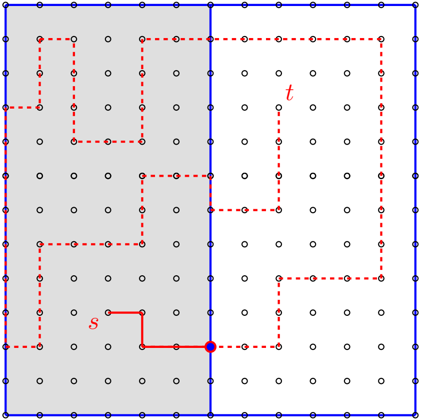

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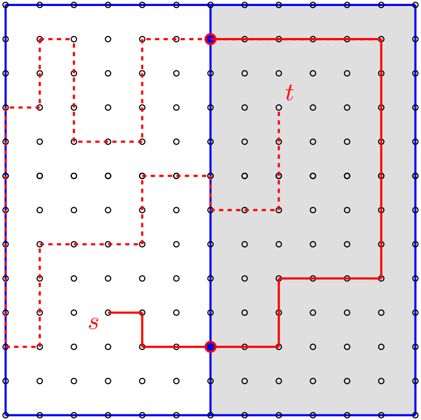

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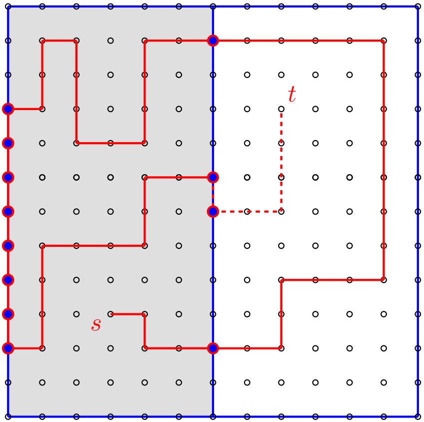

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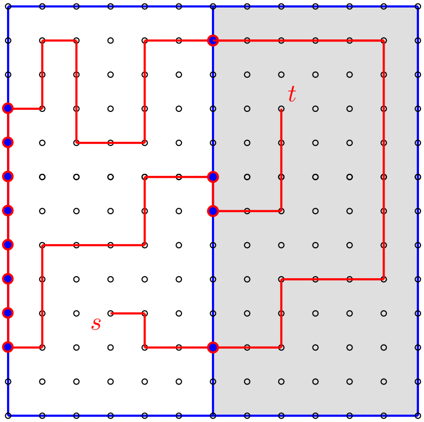

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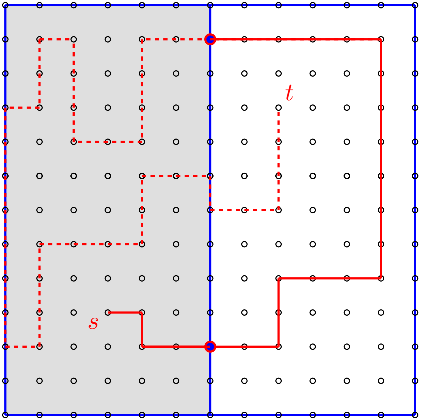

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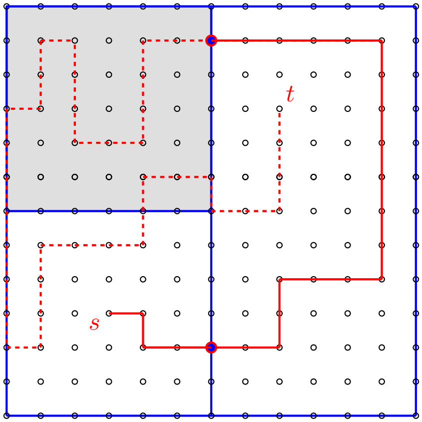

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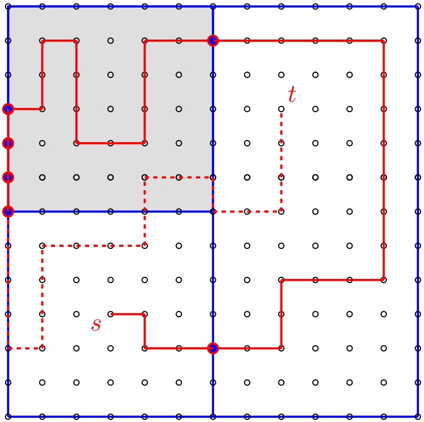

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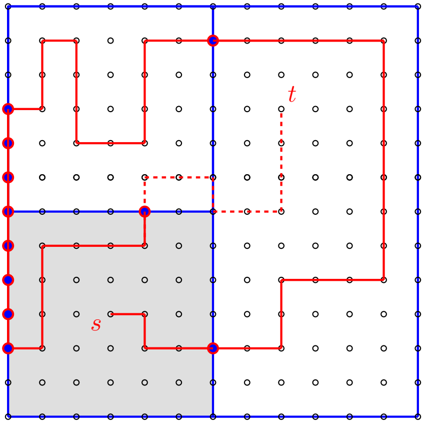

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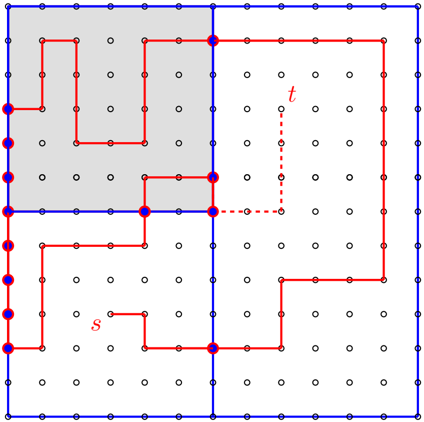
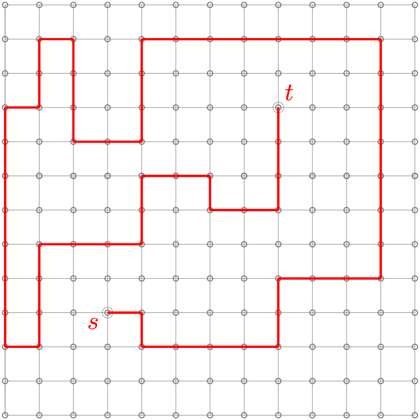


What is the cost of this approach?



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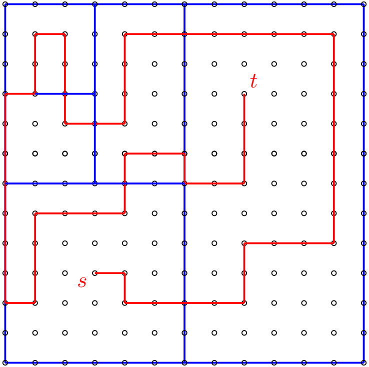
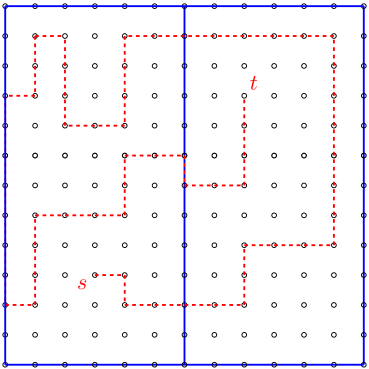


Since we maintain path weights at boundary vertices along one separating line at each level of recursion, the space cost is $O(\sqrt{n})$.

The bad news...



We need to make many (expensive) recursive calls at each level.

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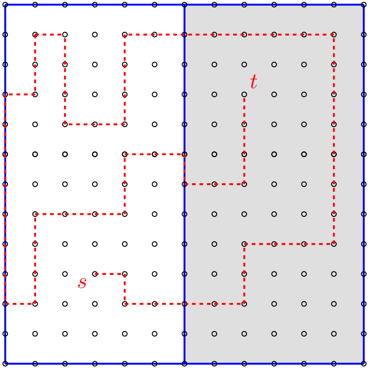

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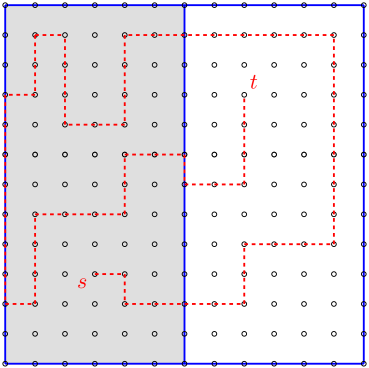

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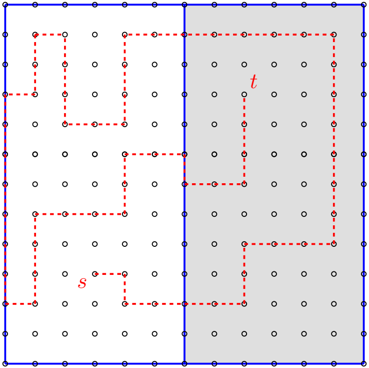

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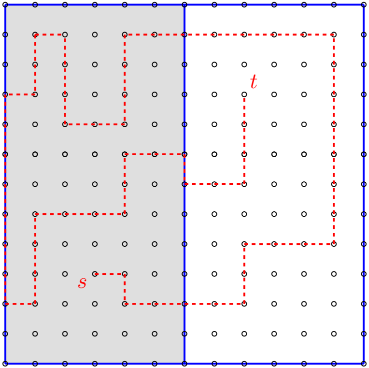

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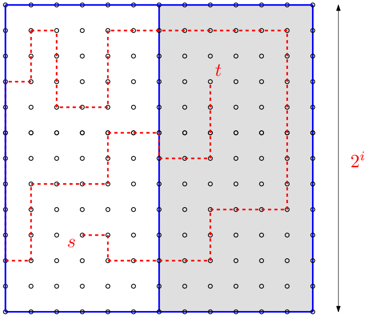
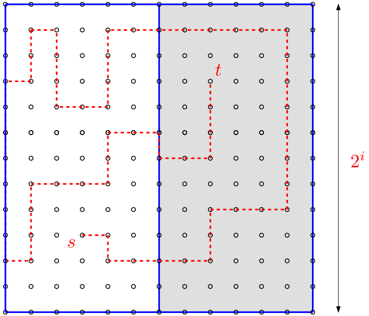


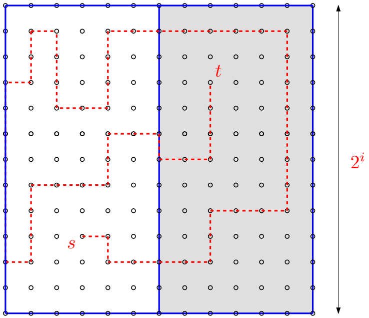
In a $2^i \times 2^i$ grid, a simple path could cross the separating line up to 2^i times. Hence, we need to make $O(2^i)$ recursive calls to subproblems at the next level.

The bad news...



Thus $\mathrm{Cost}(i)$, the cost of finding a min-weight path in a $2^i \times 2^i$ grid, satisfies $\mathrm{Cost}(i) \le 2^i \mathrm{Cost}(i-1)$,
which means $\mathrm{Cost}((\lg n)/2) = n^{O(\lg n)}$.

If we were *lucky*...



...we could *guess* the amount of time we should devote to individual recursive calls, so that we do work on a subproblem just when it will pay off...

...but we still would not be able to *certify* the solution

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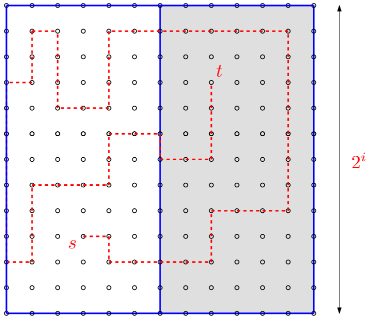

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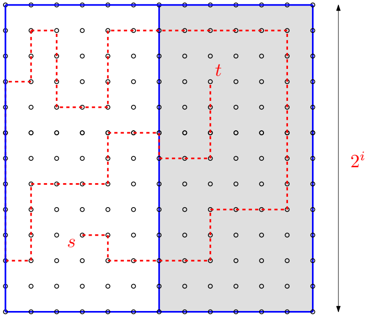
Since we can't count on being lucky...



...instead, we should construct an resource allocation scheme (*budgeted recursion*) that will be sure to subsume all possible optimal budget allocations.

Universal budget sequences...

A sequence of budgets (think bounds on the exploration length of paths) for successive subproblems at the same level of recursion is *universal* if it contains as a subsequence a sequence of budgets that is guaranteed to uncover the minimum-cost path.

- Clearly the sequence $2^{2^i}, 2^{2^i}, \ldots, 2^{2^i}$ of length 2^i is universal.
- However, we can do better...
  Consider instead the sequence σ_{2^i} defined inductively by

$$\sigma_s = \begin{cases} \langle 1 \rangle & \text{if } s = 0, \text{ and} \\ \sigma_{s-1} \diamond \langle 2^s \rangle \diamond \sigma_{s-1} & \text{otherwise,} \end{cases}$$

  (where $\diamond$ signifies concatenation of sequences) .

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- Clearly the sequence $2^{2i}, 2^{2i}, \ldots, 2^{2i}$ of length 2^i is universal.
- However, we can do better...

  Consider instead the sequence σ_{2i} defined inductively by

$$\sigma_s = \begin{cases} \langle 1 \rangle & \text{if } s = 0 \text{, and} \\ \sigma_{s-1} \diamond \langle 2^s \rangle \diamond \sigma_{s-1} & \text{otherwise,} \end{cases}$$

  (where $\diamond$ signifies concatenation of sequences) .

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Properties of this *ruler* sequence

- the sequence σ_s is computable in $O(2^s)$-time and $\tilde{O}(1)$-space;
- the sequence σ_s contains exactly 2^{s-i} appearances of the integer 2^i, for all $i \in [s]$, and nothing else;
- (universality) for any positive integer sequence $\langle d_1, \ldots, d_x \rangle$ such that $\sum_{i \in [x]} d_i \leq 2^s$, there exists a subsequence $\langle c_{i_1}, \ldots, c_{i_x} \rangle$ of σ_s such that $d_j \leq c_{i_j}$ holds for all $j \in [x]$

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Proof of universality

(By induction on s)
Suppose that $\sum_{i\in[x]} d_i \le 2^s$. Choose the smallest m such that $\sum_{i\in[m]} d_i > \frac{1}{2}\sum_{i\in[x]} d_i$. Then,
(i) by induction, both $\langle d_1, \ldots, d_{m-1}\rangle$ and $\langle d_{m+1}, \ldots, d_x\rangle$ are dominated by subsequences of σ_{s-1}, and
(ii) $d_m \le 2^s$.
Hence $\langle d_1, \ldots, d_x\rangle$ is dominated by $\sigma_s = \sigma_{s-1} \diamond \langle 2^s\rangle \diamond \sigma_{s-1}$.

Using budgeted recursion, guided by this universal; sequence...

Theorem

For any instance of the min-weight path problem on an $2^h \times 2^h$ grid the procedure determines the min-weight path in $O(2^{9h})$ time and $\tilde{O}(2^h)$ space.

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Proof sketch...

- correctness follows directly from universality property
- space complexity is clear
- The cost at the m-th level of recursion, with budget 2^s, $\mathrm{Cost}(m, 2^s)$, satisfies $\mathrm{Cost}(m, 2^s) \leq c \cdot 2^h T(2h - m, s)$, where

$$T(r, s) \; = \; \begin{cases} 2^s & \text{if } r = 0, \\ 1 + 2\sum_{0 \leq j \leq s} 2^j T(r-1, s-j) & \text{if } r > 0. \end{cases}$$

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It is straightforward to confirm that

$$T(r,s) \;=\; \begin{cases} 2^{r+1}-1 & \text{if } r>0 \text{ and } s=0,\\ 2T(r,s-1)+2T(r-1,s)-1 & \text{if } r>0 \text{ and } s>0.\end{cases}$$

Thus, $T(r,s) \le 2^{r+s+1}\binom{r+s}{s}$.
It follows that $\mathrm{Cost}(m,2^s) \le c\cdot 2^h 2^{2h-m+s+1}\binom{2h-m+s}{s}$.
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Combining the two approaches...

- Of course, it makes sense to stop the recursion when the subproblem size falls below $\sqrt{n}$.
- In fact, it pays to stop even earlier and switch to the Asano-Doerr method.
- The optimal switch point depends on the desired time-space tradeoff.

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What about *reporting* the minimum-weight path?...

- Straightforward to maintain predecessor pointer for the target vertex t, and repeat (at a multiplicative cost proportional to the optimal path *length*);
- Alternatively, we can maintain minimum path values from s and to t at *all* vertices of the top level separator. Then solve a sequence of lower-level subproblems recursively. The (time) cost is dominated by the top-level problem.
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Outline

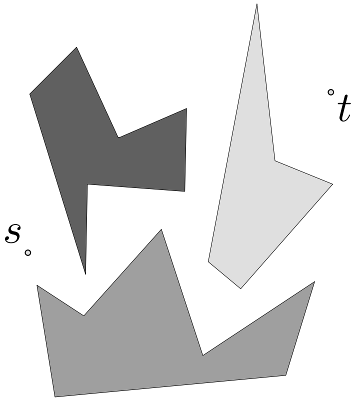
Introduction
- algorithms for shortest (min-weight) paths
- memory-constrained algorithms

Min-weight paths in grid graphs – Asano&Doerr(2011)
- overview of basic algorithm
- applying a good idea recursively

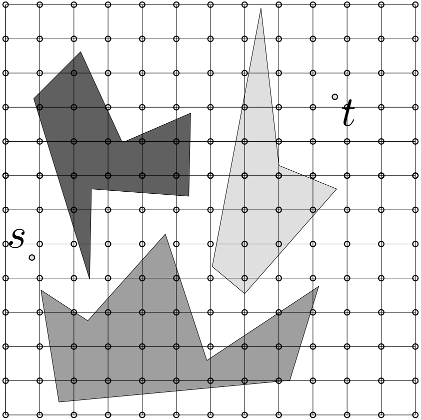
Min-weight paths in grid graphs – Refinements & Extensions
- a different recursive formulation
- budgeted recursion – exploiting a universal sequence
- combining the ideas

Beyond grid graphs...
- min-weight paths in implicit graphs
- min-weight paths in general planar graphs

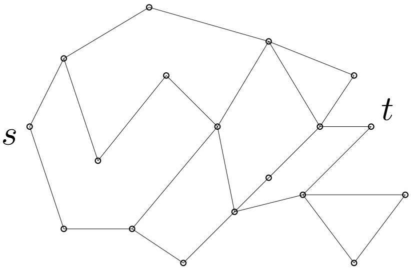
An arrangement of weighted regions with source and target...



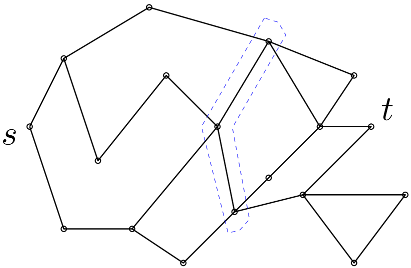
...and an overlaid grid



A planar graph



A planar graph...with a small separator



In joint work with Asano, Nakagawa and Wanatabe [MFCS 2014], this work is extended to arbitrary planar directed graphs.

Basic ideas for planar graphs...

- Use a space-efficient algorithm for constructing separators [Imai et al.]
- Maintain separators explicitly and (separated) components *implicitly* (using a representative point.
- Reconstruct triangulated components on-demand, using Reingold's log-space undirected reachability algorithm

That's it.....

And they all lived happily ever after.....

THE END

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