CS 420: Advanced Algorithm Design and Analysis
Spring 2015 – Lecture 7

Department of Computer Science
University of British Columbia

January 27, 2015

Announcements

Assignments...

- Asst3...due Thursday

Upcoming Exams / Q/A Sessions ...

- review session: Tuesday, Feb. 03, 5:30-7:00; DMPT 301
- exam: Wednesday, Feb. 04, 5:30-7:00; DMPT 301
  - covers material up to (and including) Lecture 8 (January 29)

Readings...

- material on hashing [Kleinberg, 13.6; Cormen+, chap 11; Erickson, chapt 12]
- material on closest-pair problem [Kleinberg]
- material on optimal binary search trees [Erickson 3.5, 5.6; Cormen+, chapt 13]
- material on adaptive (self-adjusting) search structures; splay trees [Erickson, chapt. 16]

Looking ahead...

Our goal, in the next few lectures is to understand how we might circumvent this lower bound, by *stepping outside the abstract comparison-based model*. We will consider:

- exploiting assumptions about the structure/size of the key space $\mathcal{U}$
- exploiting assumptions about the distribution of keys in S
- exploiting assumptions about the pattern of successive queries
- (if time permits) other issues: randomization, error tolerance...

Last class...

Assignment 1 discussion

Compact universal families $\mathcal{H}$ exist and are efficient to construct

- Kleinberg&Tardos (section 13.6) describe one construction based on modular arithmetic

Applications of universal hashing (cont.)

- finding the closest pair of points in a point set
  - intuition from 1-d: reduction to sorting
  - divide and conquer in 2-d [Kleinberg&Tardos (section 5.4)]
  - identification of *neighbourly* point pairs, using *Voronoi diagrams*

Today...

Applications of universal hashing (cont.)

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  - a randomized approach [Kleinberg&Tardos (section 13.7)]

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Dictionaries with non-uniform access patterns

- fixed (known) access frequencies
  - list-structured dictionaries
  - tree-structured dictionaries...*optimal binary search trees*

Finding the closest pair of points in a point set

Problem definition

Given a collection of n points in real d-dimensional space, identify the pair of points $\{p_i, p_j\}$whose *separation* ($||p_i - p_j||$) is smallest.

Finding the closest pair of points in a point set

A randomized incremental approach in $\Re^2$

Historical note...

- The first problem that popularized the idea of using *randomization* in the design of algorithms. [Michael Rabin, early 1970's]

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- *Las Vegas* algorithms (unlike *Monte Carlo* algorithms) use randomization to reduce the complexity of deterministic algorithms, *without compromising correctness*.
- The approach described here (and in Kleinberg) is a modification published by Golin et al., 1995.

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Finding the closest pair of points in a point set

A randomized incremental approach in $\Re^2$

Algorithm randomized closest-pair in $[0,1]^2$

re-order input points randomly: $p_1, p_2, \ldots, p_n$
$\sigma_{\min} \leftarrow \sigma(p_1, p_2)$; $i \leftarrow 3$
while $i < n+1$ **do**
    while $\mathrm{N}_{\sigma_{\min}}(p_i) \cap \{p_1, \ldots, p_{i-1}\} = \emptyset$ **do**
        $i \leftarrow i+1$
    end while
    $p_j \leftarrow$ closest point in $\{p_1, \ldots, p_{i-1}\}$ to p_i
    $\sigma_{\min} \leftarrow \sigma(p_i, p_j)$
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end while

where $\mathrm{N}_{\sigma_{\min}}(p)$ denotes the $\sigma_{\min}$-neighbourhood of p

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The algorithm proceeds in a sequence of *stages* during which the *stage invariant* $\mathrm{N}_{\sigma_{\min}}(p_i) \cap \{p_1, \ldots, p_{i-1}\} = \emptyset$ holds.

Between stages we update $\sigma_{\min}$ by computing
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The cost depends on

- the cost of testing the stage invariant
- the number and cost of stage transitions

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- by construction *no cell contains more than one point* among $\{p_1, \ldots, p_{i-1}\}$
- stage invariant fails if point p_i has a point among $\{p_1, \ldots, p_{i-1}\}$ in the *neighbourhood* of $\mathrm{cell}(p_i)$

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So the total expected cost is $O(n)$.

See Kleinberg&Tardos (Section 13.7) for full details...

Exploiting non-uniform access patterns

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- how should we organize L to minimize the *expected access cost*: $\sum_{x_i \in S} p_i \cdot \mathrm{posn}_L(x_i)$?
- simple interchange argument demonstrates that order by decreasing access probability is *optimal*

Known access probabilities for individual keys

What if our dictionary is a tree?

- how should we organize the tree T to minimize the *expected access cost*:

$$\sum_{x_i \in S} p_i \cdot (\text{depth}_T(x_i) + 1) = 1 + \sum_{x_i \in S} p_i \cdot \text{depth}_T(x_i)$$

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- how should we organize the tree T to minimize the *expected access cost*:

$$\sum_{x_i \in S} p_i \cdot (\text{depth}_T(x_i) + 1) = 1 + \sum_{x_i \in S} p_i \cdot \text{depth}_T(x_i)$$

- this tree is called an *optimal binary search tree*

Some reasonable *heuristics*...

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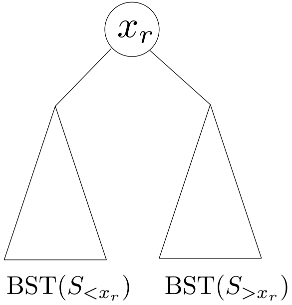
- choose root to balance the probabilities of subtrees
- put element with highest probability at the root

None of these guarantees optimal behaviour.

What does a binary search tree look like?

BST(S):



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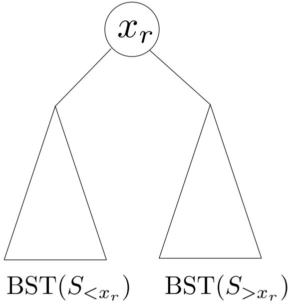
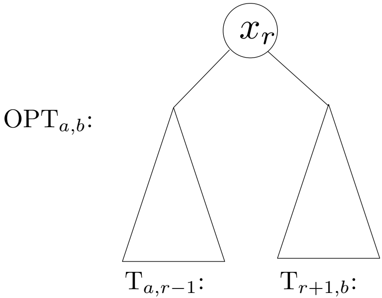


BST($\emptyset$)=NIL:

$$\text{depth}_{\text{BST}(S)}(x) = \begin{cases} 1 + \text{depth}_{\text{BST}(S_{<x_r})}(x) & \text{if } x < x_r \\ 0 & \text{if } x = x_r \\ 1 + \text{depth}_{\text{BST}(S_{>x_r})}(x) & \text{if } x > x_r \end{cases}$$

What does an *optimal* binary search tree look like?



$\mathrm{OPT}_{a,b}$ denotes the optimal tree for the key sequence $x_a, x_{a+1}, \ldots, x_b$

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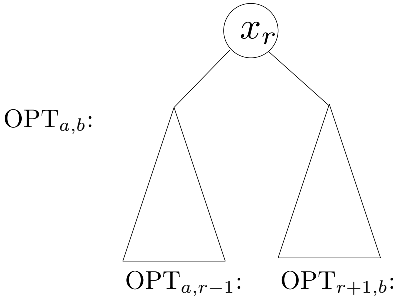

Optimal trees satisfy the *optimal substructure* property

How can we describe its cost?

Let $C_{a,b}$ denote the *cost* of $\mathrm{OPT}_{a,b}$. How does $C_{a,b}$ relate to $C_{a,r-1}$ and $C_{r+1,b}$?

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$$C_{a,b} = \begin{cases} 0 & \text{if } a > b \\ \min_{a \leq r \leq b}\{C_{a,r-1} + C_{r+1,b} + W_{a,b}\} & \text{if } a \leq b \end{cases} \tag{1}$$

where

$$W_{a,b} = \sum_{a \leq i \leq b} p_i$$

denotes the cost associated with the root node (x_r).

How can we compute its cost?

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This should suggest a different approach: *dynamic programming*

A dynamic programming solution

Algorithm pseudocode for optimal BST cost computation

```
for l = 1 to n do
    for a = 1 to n − l + 1 do
        evaluate and tabulate C_{a,a+l−1} using equation (1)
    end for
end for
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A dynamic programming solution

Algorithm pseudocode for optimal BST cost computation

1: **for** $l = 1$ to n **do**
2:     **for** $a = 1$ to $n - l + 1$ **do**
3:         evaluate and tabulate $C_{a,a+l-1}$ using equation (1)
4:     **end for**
5: **end for**

Analysis:

- each new entry $C_{a,a+l-1}$ can be computed in $O(n)$ time
- total cost ($O(n^2)$ entries) is $O(n^3)$

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- total cost ($O(n^2)$ entries) is $O(n^3)$
- Note: all $W_{a,b}$ values can be computed in $O(n)$ time!

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- the approach extends naturally to the situation where we know q_j values: the probability of accessing a key *not in the dictionary*, between x_{j-1} and x_j.
- total cost can be reduced to $O(n^2)$, by exploiting the fact that the root of $\mathrm{OPT}_{a,b}$ must lie between the root of $\mathrm{OPT}_{a,b-1}$ and the root of $\mathrm{OPT}_{a+1,b}$.

Next time...

Exploiting non-uniformity access patterns

- unknown/changing access probabilities; adaptive search structures (splay trees)