CS 420: Advanced Algorithm Design and Analysis
Spring 2015 – Lecture 16

Department of Computer Science
University of British Columbia

March 05, 2015

Announcements

Assignments...

- Asst6/7...out today (due March 19)

Midterm III...

- Q/A session...March 24; 5:30-7:00; DMPT 110
- Exam...March 25; 5:30-7:00; DMPT 110
- ...on *all* course material up to and including March 19 lecture

Announcements (cont.)

Readings...

- matchings and network flows [Kleinberg&Tardos, Chapt. 7], [Cormen et al., Chapt. 26], [Dasgupta et al., Chapter 7]
- reductions and NP-hardness [Kleinberg&Tardos, Chapt. 8, 11], [Cormen et al., Chapt. 34,35]

Last class...

Matchings and Network Flows

- matchings
  - definitions
  - bounds for bipartite matchings
  - duality

Matchings in Bipartite Graphs

Since $|M| \leq |V_L \backslash S| + |N(S)|$ holds for all matchings M and all sets $S \subseteq V_L$, it follows that:

Claim:

$$\max_{\text{matchings } M} |M| \leq \min_{S \subseteq V_L} \{|V_L \backslash S| + |N(S)|\}$$

Note: this holds even if edges can be chosen *fractionally*

Matchings in Bipartite Graphs – Berge's Theorem

Suppose that M does not admit an augmenting path.

Let S_M denote the set of vertices in $v \in V_L$ such that there exists an even length alternating path to v from some unsaturated vertex in V_L. Then

1. every vertex $v \in V_L \backslash S_M$ is saturated (otherwise a path of length 0 exists)
2. every vertex $w \in N(S_M)$ is saturated (otherwise an augmenting path to w exists)
3. no edge (v, w) of M joins $V_L \backslash S_M$ to $N(S_M)$ (otherwise v should belong to S_M)

Thus... $$|M| \geq |V_L \backslash S_M| + |N(S_M)|$$

Matchings in Bipartite Graphs

Taken together with the earlier **Claim** this proves the following:

Theorem[Kőnig 1931]

$$\max_{\text{matchings } M} |M| = \min_{S \subseteq V_L} \{|V_L \setminus S| + |N(S)|\}$$

Matchings in Bipartite Graphs

Corollaries

1. There is an efficient algorithm to construct a maximum matching in a bipartite graph —search for augmenting paths
2. The algorithm is *self-certifying* —the set S_M provides a *certificate* of the optimality of M
3. The set $V_L \backslash S \cup N(S)$ is a *vertex cover* of G. The Theorem establishes the fact that, in bipartite graphs,

$$\max_{\text{matchings } M} |M| = \min_{\text{vertex covers } C} \{|C|\}$$

Today...

Matchings and Network Flows

- network flows
  - definitions
  - relationship with bipartite matchings
  - duality

Coming up...

Reductions and relative hardness of problems

- reductions
  - definitions
  - role(s) in establishing relative hardness
  - examples (review)
- overview of problems with efficient algorithms
  ... and related problems with no known efficient algorithm
- the complexity classes **P** and **NP**
- **NP**-hardness and **NP**-completeness

Network Flows

Definitions

A *capacitated network* is a directed graph G with

1. two distinguished vertices: s (the *source*) and t (the *sink*); and
2. a non-negative number $c(e)$, associated with each edge e, called the *capacity* of e

A *flow* (from s to t) in G is a function $f : E \rightarrow \Re$ that satisfies:

1. (*capacity constraints*) $0 \leq f(e) \leq c(e)$, for all $e \in E$; and
2. (*flow conservation*) $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$.

The *value* of flow f, denoted $|f|$, is defined as:

$$|f| = \sum_{e \text{ out of } s} f(e)$$

The maximum-flow problem

Given a capacitated network G, find a flow from s to t of maximum value

Network Flows

The maximum-flow problem

As with matchings we can ask:

1. can we bound the value of the maximum flow by some natural property of the network?
2. given a flow, how can we improve (augment) it to increase its value?
3. how will we know when we are done?

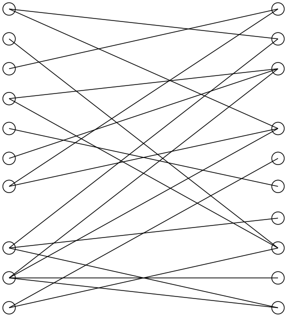
Network Flows and Bipartite Matchings

Maximum bipartite matching can be *reduced* to maximum flow

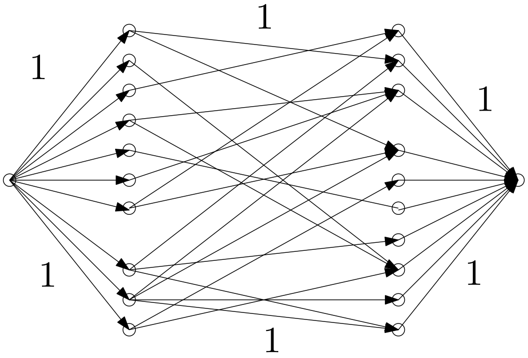
Given a bipartite graph G, with vertex bi-partition V_L and V_R:

1. direct all edges of E from V_L to V_R and assign each one capacity 1
2. create a new source vertex s and sink vertex t
3. add an edge with capacity 1 from s to each vertex in V_L; and
4. add an edge with capacity 1 from each vertex in V_R to t

Network Flows and Bipartite Matchings



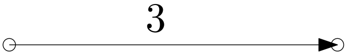
Network Flows and Bipartite Matchings



Network Flows and Bipartite Matchings

Maximum flow in *integer* capacitated networks can be *reduced* to maximum flow in *unit* capacitated networks

replace



Network Flows and Bipartite Matchings

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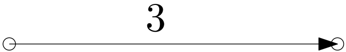

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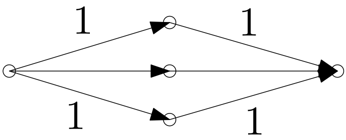
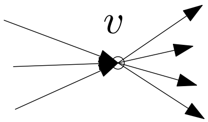


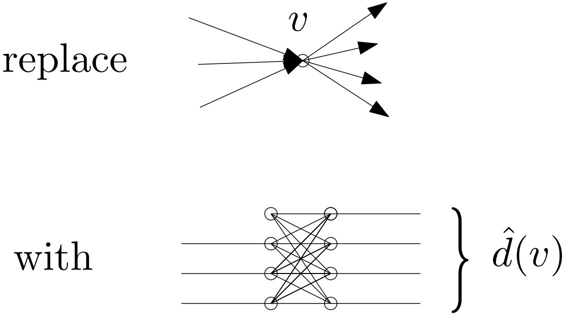
Network Flows and Bipartite Matchings

Maximum flow in *unit* capacitated networks can be *reduced* to maximum bipartite matching

replace 

Network Flows and Bipartite Matchings

Maximum flow in *unit* capacitated networks can be *reduced* to maximum bipartite matching



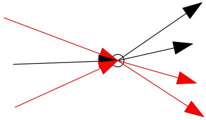
Network Flows and Bipartite Matchings

Claim The resulting bipartite graph has a matching of size $f + \sum_v \hat{d}(v)$ if and only if the original network had a flow of value f.

Network Flows and Bipartite Matchings

Matching reflects the flow through a vertex

replace



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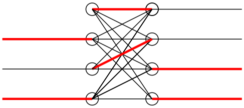
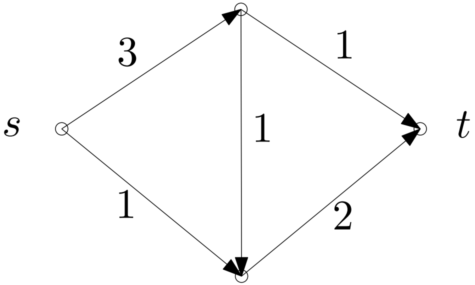


Network Flows and Bipartite Matchings

An example of the full reduction...



Network Flows and Bipartite Matchings

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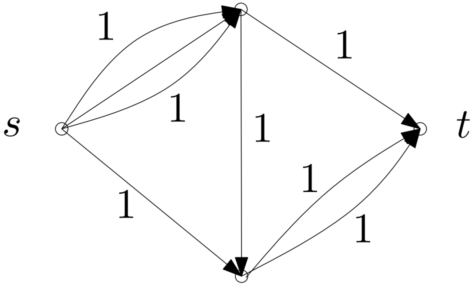

Network Flows and Bipartite Matchings

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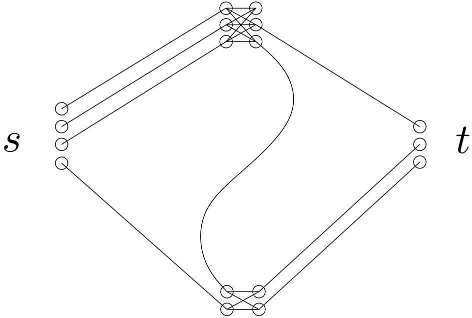
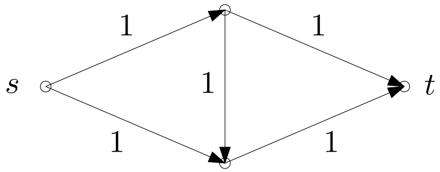


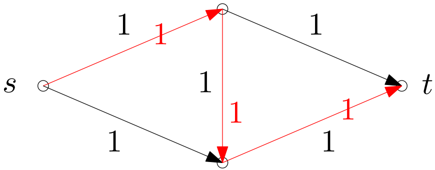
Network Flows and Bipartite Matchings

A capacitated network...



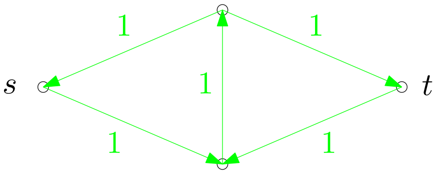
Network Flows and Bipartite Matchings

...and a flow



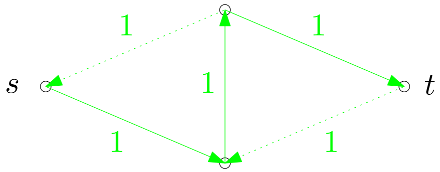
Network Flows and Bipartite Matchings

The residual flow graph...



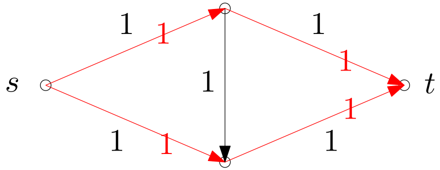
Network Flows and Bipartite Matchings

and associated *augmenting path*



Network Flows and Bipartite Matchings

The augmented flow... (note edge-disjoint paths)



Two Applications...

- resilience of sensor networks
- can the Canucks make the playoffs?

Reductions and relative hardness of problems

We will write A $\leqslant$ B to denote the fact that problem A is *reducible to* problem B. Informally, this means

1. a subroutine for solving problem B can be used as a *black box* in solving problem A
2. instances of problem A can be transformed to instances of problem B in such a way that a solution to the latter can be transformed back into a solution of the former

Reductions and relative hardness of problems

We have seen many examples throughout the course...

- element-distinctness $\lessapprox$ closest-pair $\lessapprox$ sorting
- transitive-closure $\lessapprox$ Boolean-matrix-product
- all-pairs-shortest-paths-with-arbitrary-weights
  $\lessapprox$ all-pairs-shortest-paths-with-non-negative-weights
- edit-distance (sequence-alignment)
  $\lessapprox$ single-source-shortest-path

Reductions and relative hardness of problems

We have seen many examples throughout the course...

- bipartite-matching $\leq$ bipartite-vertex-cover $\leq$ bipartite-matching
- bipartite-matching $\leq$ unit-capacitated-network-flow $\leq$ bipartite-matching
- integer-capacitated-network-flow $\leq$ unit-capacitated-network flow

Reductions and relative hardness of problems

and in homework assignments...

- minimum-colour-transition-path $\leq$ min-cost-shortest-path
- vertex-cover $\leq$ minimum-colour-path

Coming up...

Reductions and relative hardness of problems

- overview of problems with efficient algorithms
  ... and related problems with no known efficient algorithm
- reductions...treated more formally
- the complexity classes **P** and **NP**
- **NP**-hardness and **NP**-completeness

Coming up...

Reductions and relative hardness of problems

- reductions
  - definitions
  - role(s) in establishing relative hardness
  - examples (review)
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