# CS 420: Advanced Algorithm Design and Analysis
# Spring 2015 – Lecture 15

Department of Computer Science
University of British Columbia

March 03, 2015

# Announcements

## Assignments...

- Sample solutions to Asst 5 have been posted
- Asst6...out Thursday

## Midterm II...

- Q/A session...today; 5:30-7:00; DMPT 310
- Exam...tomorrow (March 04); 5:30-7:00; DMPT 310
- ...on *all* course material up to and including last Thursday's class

# Announcements (cont.)

Readings...

- minimum-cost path problems [Erickson, Chapt 21, 22; Cormen+, Chapt 25 26; ...]
- edit-distance [Erickson, Chapt. 5.5, 6; Kleinberg&Tardos, Chapt. 6.6 & 6.7]
- Goldberg et al,. "Efficient point-to-point shortest path algorithms"
- matchings and network flows [Kleinberg&Tardos, Chapt. 7], [Cormen et al., Chapt. 26], [Dasgupta et al., Chapter 7]

# Last class...

Edit distance problems (cont.)

- reformulation as (single source, single destination) min-cost path problem
- solution by reweighted Dijkstra

Min-cost path problems (cont.)

- Dijkstra modifications for single-source single-destination problems
  - bi-directional Dijkstra
  - goal-directed Dijkstra

# Today...

## Matchings and Network Flows

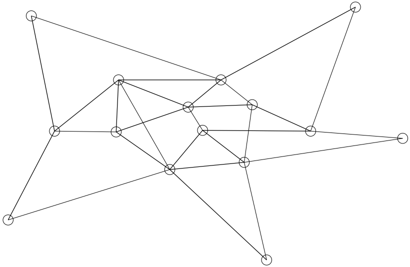
- matchings
  - definitions
  - bounds for bipartite matchings
  - duality

# Next time...

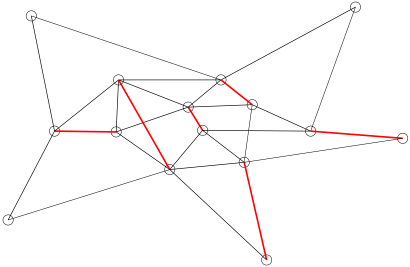
## Matchings and Network Flows

- network flows
  - definitions
  - relationship with bipartite matchings
  - duality

# Matchings in Graphs

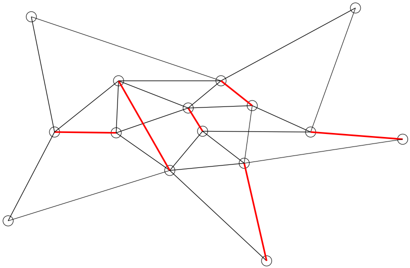


# Matchings in Graphs



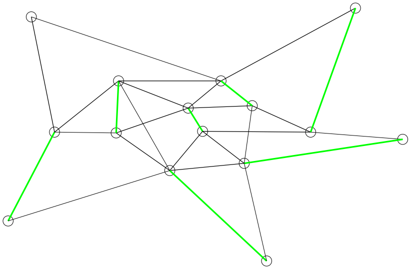
A *matching* in undirected graph $G$ is a subset $M \subseteq E$ such that no two edges of $M$ share an endpoint.

# Matchings in Graphs



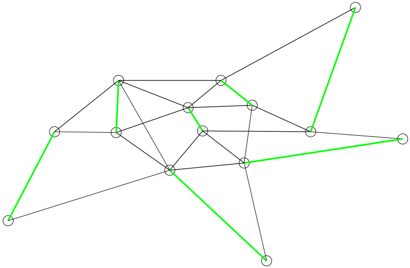
A matching is *maximal* if it is not a proper subset of a larger matching
(i.e. no pair of *unsaturated* vertices share an edge).

# Matchings in Graphs



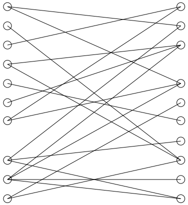
A matching is *maximum* if it has maximum cardinality among all matchings of $G$.

# Matchings in Graphs



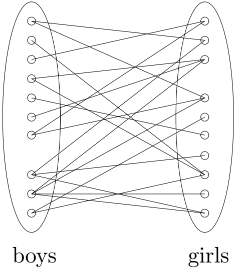
A matching is *perfect* if it saturates all vertices of $G$ (i.e. it has cardinality $|V|/2$).

# Matchings in Bipartite Graphs



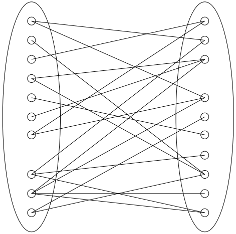
A particularly interesting case arises when the underlying graph is *bipartite*.

# Matchings in Bipartite Graphs




A matching is a selection of *compatible* pairs...

# Matchings in Bipartite Graphs

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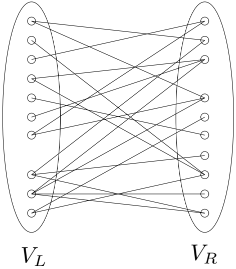
# Matchings in Bipartite Graphs

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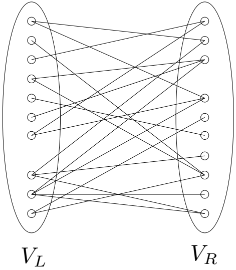
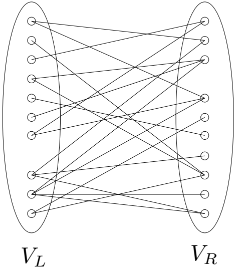
# Matchings in Bipartite Graphs



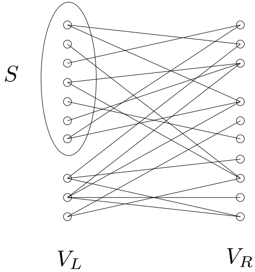

Are there properties of $G$ that allow us to *bound* the size of the maximum matching?

# Matchings in Bipartite Graphs



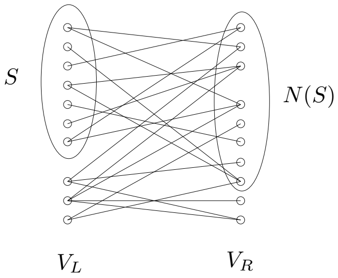

It certainly can't be larger than $\min\{|V_L|, |V_R|\}$....
but, more generally,...

# Matchings in Bipartite Graphs



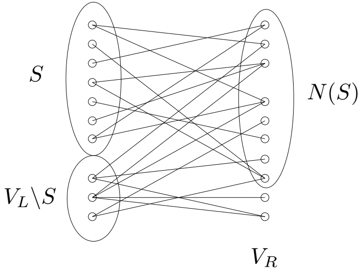

If we choose an *arbitrary* subset $S \subseteq V_L$...

# Matchings in Bipartite Graphs



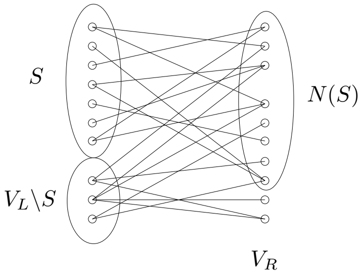

If we choose an *arbitrary* subset $S \subseteq V_L$... this distinguishes a subset $N(S) \subseteq V_R$, the *neighbours* of $S$

# Matchings in Bipartite Graphs




Every edge in a matching $M$ must have a left endpoint in $V_L \backslash S$, or a right endpoint in $N(S)$, or both. WHY?

# Matchings in Bipartite Graphs




So...

$$|M| \leq |V_L \backslash S| + |N(S)|$$

# Matchings in Bipartite Graphs

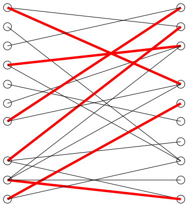
Since this holds for all matchings $M$ and all sets $S \subseteq V_L$, we have shown:

**Claim:**

$$\max_{\text{matchings } M} |M| \leq \min_{S \subseteq V_L} \{|V_L \backslash S| + |N(S)|\}$$

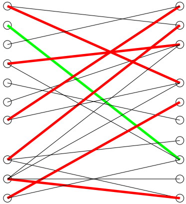
Note: this holds even if edges can be chosen *fractionally*

# Matchings in Bipartite Graphs



Suppose that we are given a matching $M$, how can we improve it?

# Matchings in Bipartite Graphs



We can add an edge that joins unsaturated vertices.

# Matchings in Bipartite Graphs

More generally, we can find an *alternating path* $P$ joining two unsaturated vertices (an *augmenting path*)

$P$ : 

# Matchings in Bipartite Graphs

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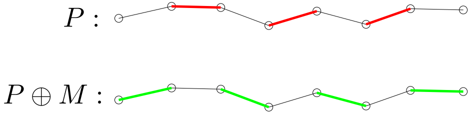


and *augment* with respect to $P$: replace $M$ with $M \oplus P$ ($\oplus$ denotes "symmetric difference")

# Matchings in Bipartite Graphs

Each augmentation with respect to an augmenting path increases the size of the matching by one.

What happens if there are no augmenting paths?

Then the current matching is a maximum matching! (That is, every improvable matching must admit an augmenting path.)

# Matchings in Bipartite Graphs – Berge's Theorem

Suppose that $M$ does not admit an augmenting path.

Let $S_M$ denote the set of vertices in $v \in V_L$ such that there exists an even length alternating path to $v$ from some unsaturated vertex in $V_L$. Then

1. every vertex $v \in V_L \backslash S_M$ is saturated (otherwise a path of length 0 exists)
2. every vertex $w \in N(S_M)$ is saturated (otherwise an augmenting path to $w$ exists)
3. no edge $(v, w)$ of $M$ joins $V_L \backslash S_M$ to $N(S_M)$ (otherwise $v$ should belong to $S_M$)

Thus... $$|M| \geq |V_L \backslash S_M| + |N(S_M)|$$

# Matchings in Bipartite Graphs

Taken together with our earlier **Claim**:

$$\max_{\text{matchings } M} |M| \leq \min_{S \subseteq V_L} \{|V_L \backslash S| + |N(S)|\}$$

this proves the following:

**Theorem**[Kőnig 1931]

$$\max_{\text{matchings } M} |M| = \min_{S \subseteq V_L} \{|V_L \backslash S| + |N(S)|\}$$

# Matchings in Bipartite Graphs

**Corollaries**

1. There is an efficient algorithm to construct a maximum matching in a bipartite graph
   —search for augmenting paths
2. The algorithm is *self-certifying*
   —the set $S_M$ provides a *certificate* of the optimality of $M$
3. The set $V_L \backslash S \cup N(S)$ is a *vertex cover* of $G$.
   The Theorem establishes the fact that, in bipartite graphs,

$$\max_{\text{matchings } M} |M| = \min_{\text{vertex covers } C} \{|C|\}$$

# Coming up...

## Matchings and Network Flows

- network flows
  - definitions
  - relationship with bipartite matchings
  - duality