Efficient Mechanism Design
Bandwidth Allocation in Computer Network

Presenter: Hao MA

Game Theory Course Presentation
April 1st, 2014
Efficient Mechanism Design focus on the mechanism that lead to efficient allocation!

Quick-fire Question
Price of Anarchy?
**Price of Anarchy (PoA):** PoA is a measure of the extent to which system efficiency degrades due to selfish behaviour of its agents.

Define $s$ as a strategy profile, $S$ as the set of all strategy profiles and $E \subseteq S$ is the set of strategies in equilibrium.

For Welfare function $W$ / Cost function $C$.

$$\text{PoA} = \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)} = \frac{\max_{s \in E} C(s)}{\min_{s \in S} C(s)}$$

*Note: PoA $\geq 1$, and the smaller, the better.*
**Price of Anarchy (PoA):** PoA is a measure of the extent to which system efficiency degrades due to selfish behaviour of its agents.

Define \( s \) as a strategy profile, \( S \) as the set of all strategy profiles and \( E \subseteq S \) is the set of strategies in equilibrium.

For Welfare function \( W \) / Cost function \( C \).

\[
PoA = \frac{\max_{s \in S} W(s)}{\min_{s \in E} W(s)} = \frac{\max_{s \in E} C(s)}{\min_{s \in S} C(s)}
\]

*Note: \( PoA \geq 1 \), and the smaller, the better.*
Figure: Pigou’s example: selfish routing problem.
Name: Steve
Position: CEO of a big Internet Provider

Personality:

- Cares more about the **best use** of network resources (efficient allocation) than **money** in his pocket (revenue maximization)
- No price discrimination, and charge each user **the same price** for network resource per unit.
Name: Steve
Position: CEO of a big Internet Provider

Personality:

- Cares more about the best use of network resources (efficient allocation) than money in his pocket (revenue maximization)
- No price discrimination, and charge each user the same price for network resource per unit.
Name: Steve
Position: CEO of a big Internet Provider
Personality:
• Cares more about the **best use** of network resources (efficient allocation) than **money** in his pocket (revenue maximization)
• No price discrimination, and charge each user **the same price** for network resource per unit.
multiple smaller Internet Providers
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
- User $r$ gets capacity $d_r$.
- User $r$ receives a utility $U_r(d_r)$
- $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

\[
\text{maximize } \sum_r U_r(d_r) \tag{1}
\]

subject to:

\[
\sum_r d_r \leq C;
\]

\[
d_r \geq 0, \ r = 1, \ldots, R.
\]
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
  - User $r$ gets capacity $d_r$.
  - User $r$ receives a utility $U_r(d_r)$
  - $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\text{maximize } \sum_r U_r(d_r)$$  \hspace{1cm} (1)

subject to:

$$\sum_r d_r \leq C;$$

$$d_r \geq 0, \ r = 1, \ldots, R.$$
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
- User $r$ gets capacity $d_r$.
  - User $r$ receives a utility $U_r(d_r)$
  - $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\text{maximize } \sum_r U_r(d_r)$$

subject to:

$$\sum_r d_r \leq C;$$

$$d_r \geq 0, r = 1, \ldots, R.$$
Problem Formulation

• A communication link of capacity $C > 0$
• $R$ users
• User $r$ get capacity $d_r$.
• User $r$ receives a utility $U_r(d_r)$
  • $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\text{maximize} \sum_r U_r(d_r)$$

subject to:

$$\sum_r d_r \leq C;$$

$$d_r \geq 0, \ r = 1, \ldots, R.$$
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
- User $r$ gets capacity $d_r$.
- User $r$ receives a utility $U_r(d_r)$
- $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\begin{align*}
\text{maximize} & \quad \sum_r U_r(d_r) \\
\text{subject to} & \quad \sum_r d_r \leq C; \\
& \quad d_r \geq 0, \quad r = 1, \ldots, R.
\end{align*}$$

(1)
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
- User $r$ gets capacity $d_r$.
- User $r$ receives a utility $U_r(d_r)$
- $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\text{maximize } \sum_r U_r(d_r)$$

subject to:

$$\sum_r d_r \leq C;$$

$$d_r \geq 0, r = 1, ..., R.$$
Problem Formulation

- A communication link of capacity $C > 0$
- $R$ users
- User $r$ gets capacity $d_r$.
- User $r$ receives a utility $U_r(d_r)$
- $U_r(d_r)$ is concave, strictly increasing and continuously differentiable with domain $d_r > 0$

Given a complete knowledge and centralized control of the system, the optimization problem becomes

$$\text{maximize } \sum_r U_r(d_r)$$

subject to:

$$\sum_r d_r \leq C;$$

$$d_r \geq 0, \ r = 1, \ldots, R.$$
Problem?
Problem?

Utility functions are not available to the manager.

What should Steve do?
Utility functions are not available to the manager.

What should Steve do?
Suggested Mechanism

Proportional Allocation Mechanism: Each user $r$ gives a payment $w_r$ ($w_r \geq 0$) to Steve. Given the vector $w = (w_1, \cdots, w_r)$, Steve chooses a capacity allocation $d = (d_1, \cdots, d_r)$. Each user is charged with the same price $\mu > 0$, leading to $d_r = \frac{w_r}{\mu}$.

$$\sum_r \frac{w_r}{\mu} = C \Rightarrow \mu = \frac{\sum_r w_r}{C}$$

Quick-fire Question
Suggested Mechanism

Direct?
Logic flow of the analysis

- Price-taking Agent Model: Users do not anticipate the effect of their actions on the prices of the link per unit ($\mu$), and they consider the price to be fixed and they select the best declarations $w_r$ given $\mu$.

↓ relaxation

- Price-Anticipating Agent Model: Users can anticipate the effects of their actions.
Proportional Allocation Mechanism: Price-taking Agent Model

Given a price $\mu > 0$, user $r$ try to maximize its payoff function for $w_r \geq 0$:

$$P_r(w_r; \mu) = U_r \left( \frac{w_r}{\mu} \right) - w_r \quad (Quasilinear in w_r)$$

A pair $(w, \mu)$ is a competitive equilibrium if users maximize their payoff

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \quad \forall \hat{w}_r \geq 0, \ r$$

[Kelly 2007] shows that when users are price-takers, there exists a competitive equilibrium, and the resulting allocation solves the optimization problem (1)
Proportional Allocation Mechanism: Price-taking Agent Model

Given a price $\mu > 0$, user $r$ try to maximize its payoff function for $w_r \geq 0$:

$$P_r(w_r; \mu) = U_r \left( \frac{w_r}{\mu} \right) - w_r \ (Quasilinear \ in \ w_r)$$

A pair $(w, \mu)$ is a competitive equilibrium if users maximize their payoff

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \ \forall \hat{w}_r \geq 0, \ r$$

[Kelly 2007] shows that when users are price-takers, there exists a competitive equilibrium, and the resulting allocation solves the optimization problem (1)
Theorem

[KELLY 1997] Assume that for each user $r$, the utility function $U_r$ is concave, strictly increasing, and continuously differentiable. Then there exists a competitive equilibrium, i.e., a vector $w = (w_1, \cdots, w_r) \geq 0$ and a scalar $\mu > 0$ satisfying

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \quad \forall \hat{w}_r \geq 0, \ r$$

$$\mu = \frac{\sum_r w_r}{C}$$

In this case, the scalar $\mu$ is uniquely determined, and the vector $d = \frac{w}{\mu}$ is a solution to the optimization problem (1). If the functions $U_r$ are strictly concave, then $w$ is uniquely determined as well.
Proof

Step 1: Aim: Find the equivalent/optimality condition for the competitive equilibrium.

Given $\mu > 0$, w satisfy

$$P_r(w_r; \mu) \geq P_r(\hat{w}_r; \mu) \quad \forall \hat{w}_r \geq 0, \ r$$

if and only if

$$\frac{dP_r(w_r; \mu)}{dw_r} = 0 \quad \text{if } w_r > 0$$
$$\frac{dP_r(0; \mu)}{dw_r} \leq 0 \quad \text{if } w_r = 0$$

($P_r$ is also concave) namely

$$\dot{U}_r \left( \frac{w_r}{\mu} \right) = \mu \quad \text{if } w_r > 0$$
$$\dot{U}_r(0) \leq \mu \quad \text{if } w_r = 0$$
Proof

Step 2: Aim: There exists a $d$ that satisfies constraints of similar form.

What We know: at least one optimal solution to the optimization problem exists (Why?)

Lagrangian:

$$L(d, \mu) = \sum_r U_r(d_r) - \mu \left( \sum_r d_r - C \right)$$

Slater constraint qualification $\sqrt{\Rightarrow}$ existence of $\mu \sqrt{\Rightarrow}$ so the optimal $d$ will satisfy

$$U_r'(d_r) = \mu \text{ if } d_r > 0$$
$$U_r'(0) \leq \mu \text{ if } d_r = 0$$

$$\sum_r d_r = C.$$ 

There exists a pair $(d, \mu)$ that satisfy the constraints above, and $\mu$ is unique and $\mu > 0$.

Quick-fire Question
Proof

- Step 3: If the pair \((d, \mu)\) satisfies constraint in Step 2, let \(w = \mu d\) and \((w, \mu)\) will satisfy the constraint in Step 1 (i.e. competitive equilibrium).

- Step 4: If \(w\) and \(\mu > 0\) satisfy constraint in step 1 (i.e. competitive equilibrium), let \(d = \frac{w}{\mu}\), and \((d, \mu)\) will satisfies constraints in Step 2.

- Step 5: Complete the proof.
Proportional Allocation Mechanism: Price-Anticipating Agent Model

Now the agents know that they can affect the price!

It is possible to show that a Nash equilibrium exists and that is unique.

**Theorem**

[Johari 2004] Let $R \geq 2$, let $d^{CE}$ be an allocation profile achievable in competitive equilibrium and let $d^{NE}$ be the unique allocation profile achievable in Nash equilibrium. Then any profile of valuation functions $U_r$ for which $\forall r, U_r(0) \geq 0$ satisfies

$$\sum_r U_r(d^{NE}_r) \geq \frac{3}{4} \sum_i U_r(d^{CE}_r).$$

**Quick-fire Question**
Proportional Allocation Mechanism: Price-Anticipating Agent Model

In other words, the price of anarchy is \( \frac{4}{3} \). Even in the worst case, the strategic behaviour by agents will only cause a small reduction in social welfare.
Proportional Allocation Mechanism: Price-Anticipating Model

Something Else:

- Not bad!
- It achieves minimal price of anarchy, as compared to a broad family of mechanisms in which
  - agents’ declarations are a single scalar;
  - the mechanism charges all users the same rate.
- When mechanism is allowed to charge users at different prices, a VCG-like mechanism can be used to achieve full efficiency.
In a game where users of a congested single resource anticipate the effect of their actions on the price of the resource, the aggregate utility received by the users is at least $\frac{3}{4}$ of the maximum possible aggregate utility.
References

Johari 2004  

Kelly 1997  

Shoham 2009  

Vijay 1998  
Question?