Theoretical Analysis of the Extension of VCG Equilibrium in Sponsored Search Auction

Course Project Game Player 2

University of British Columbia

April 24, 2013

Abstract

In the sponsored search auction, VCG and GSP have been compared by researchers from computer science and economics in the last few years. However, there have been some experiments not well consistent with the existing literatures. We proposed that an assumption could be refined, namely the dominant-strategy equilibrium is still the unique valid VCG equilibrium concept in the sponsored search auction scenario. In this paper, we explored an approach to extend the VCG equilibrium concept by applying locally envy free Nash equilibrium (LEFNE) from GSP. We achieved two main results by theoretical analysis, which are 1) efficiency is still guaranteed under LEFNE in VCG; 2) the range of revenue of VCG and GSP coincide with each other. Thus, our proposed extension is supported to be valid.

1 INTRODUCTION

When most people hear the word auction, they may not think it happens almost every day in our lives conducted by ourselves. Actually, rather than trading ceramic art work like a Summer Palace bronze head from China, selling Tunas in a fish market in Tokyo, or several energy companies bidding for extracting natural gas, the online advertising auctions happen million times a day when people enter some queries into a web search engine. You may have benefited from or been annoyed by the ads displayed around the result pages when using a search engine. However one of the primary sources of a search engine company's revenue is by selling the positions (slots) through sponsored search auction (position auction).

Someone may question that how many users would actually click those ads. Hal R. Varian claimed in [1] that a well-designed advertisement would get 3% click-through-rate [1], and the conversion rate [2] of a typical landing page linked

[1]CTR, the estimated number of clicks in a certain period of time or the probability of clicking the ad when it is shown.

[2]The probability of buying, downloading, signing up or taking any action when the user browse a website.

from the ads is also around 3%. This means only less than 1‰ of users who see the ads actually take an action desired by the advertisers. Though the rate seems quite low, it is much more effective than advertising in other traditional media concerning the dramatically large daily queries online. Not surprisingly, sponsored search auction has contributed billions of revenues for search engine companies in the last decade.

The most widely used process of sponsored search auction works as follows. When a specific keyword is queried by the users, the advertisers doing business related to it hope to display their ads. There are only a few available positions to display, e.g., currently Google has 8 available positions on the right and 3 on the top of the returned search results. Every participating advertiser submits a bid for a keyword when the auction is run[3]. The ads are allocated with different positions or not able to be displayed according to the bids from all advertisers. They will only need to pay when the users click their ads. Figure 1 shows an example of searching "shoe canada" in Google. We can see that the ad of website *shoeme.ca* is placed in the best position by Google Ad Auction system, at the same time it is also the first one in the returned search results, which makes it interesting to think about the advertising strategy of this company.

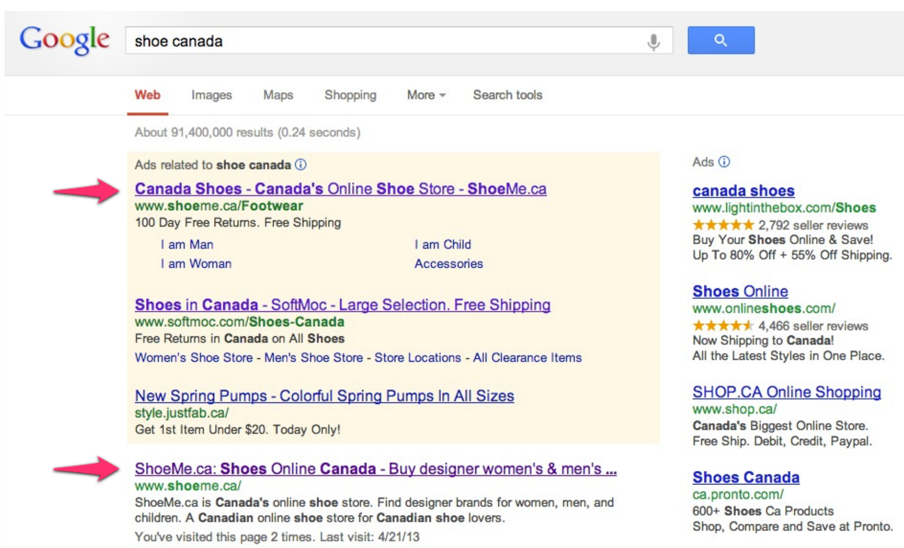

Figure 1: An example of the searching "shoe canada"

The Generalized Second Price auction (GSP) and its variants currently are the most widely used mechanisms by many web search companies. It evolved from the Generalized First Price auction (GFP). In GFP the advertisers are

[3]We assume that the bid is less than or equal to the advertiser's value-per-click i.e. the expected value for advertiser from each of user's clicks through the ad to the landing website, which can be estimated by analytical tools like Good Analytics.

allocated with the positions in a descending order of the submitted bids, e.g. the ad with the highest bid get the best position, and that with the second highest get the second best position, and so on. Then the advertiser paid the amount equal to her bid for one click of her ads. The ad auction designer soon found that this auction was not attractive since the advertisers would like to reduce their bids as low as possible while retain their positions[1]. In addition, this lead to a significantly heavy burden on servers due to the constant system monitoring. To cope with these problems, designers decided to improve the system by directly charging the advertisers the next highest bid since that's what the advertisers would like to do. This was actually just a decision made by the engineers of search engine company, without anything to do with the theoretically-well-understood Vickrey auction. Thus it is generally believed the theoretical analysis and practical use of GSP have been developed in a parallel fashion.

Vivkrey-Clarke-Groves (VCG) mechanism is an alternative auction with a couple of good theoretical properties. It has a dominant-strategy equilibrium of truth telling, i.e., each advertiser should bid her true value-per-click. In this equilibrium, the allocation of positions is efficient. [2] shows that VCG maximizes the auctioneer's revenue within the space of efficient mechanisms. In the case of single-position auction, VCG mechanism and GSP mechanism are equivalent, while in the multiple-position cases they are not. With all these properties, VCG is a nice alternative choice in sponsored search auction with the benefits like reducing the problem of considering the advertiser's strategic bidding behaviours and improving the auctioneer's revenue. However, VCG is not used in practice[4]. As Hal R. Varian, the Chief Economist of Google, said, actually the auction designers were not aware of VCG when designing the earlier versions of Google Ad Auction. Moreover, no other big search engine companies has ever claimed the use of VCG mechanism for ad auction in public. Therefore, it is unknown that if using VCG for search engine ad auction in practice is appropriate.

There have been considerably many literatures discussing the difference and similarities between these two mechanisms. Edelman *et al.* [4] and Varian [5] performed nice theoretical analysis for GSP independently. Since there is no dominant-strategy equilibrium in GSP, Edelman *et al.* and Varian proposed the concepts of *Locally Envy Free Nash Equilibrium* (LEFNE) and *Symmetric Nash Equilibrium* (SNE) respectively. These two concepts have been proved to be equivalent. Efficiency can be achieved in this equilibrium of GSP. They also proved that the revenue made by VCG is the lower bound of the revenues that can be made by GSP, under the assumption that a set of LEFNFs are achieved in GSP and dominant-strategy equilibrium in VCG.

So far, in some existing literatures there are two observations not consistent

[4]Recently Facebook has employed VCG for their Ad Auction system [3].

with the theoretical analysis. First, the unique dominant-strategy equilibrium in VCG is such a strong one, and it is the only equilibrium considered, however a number of experiments have observed that it is rarely achieved ([6], [7]). Second, the revenue made by GSP were claimed to be more than or equal to that made by VCG, but some experiments report that GSP doesn't beat VCG clearly and consistently in terms of revenue. In this paper, I will investigate the idea of extending the equilibrium concept in VCG by applying LEFNE in the specific sponsored search auction, which enables us to analyze the above two problems. Some theoretical analysis will be performed to check whether this extension is valid.

This paper is organized as follows. In section 2, I will introduce the a model used for sponsored search auction and the mechanisms of GSP and VCG under this model. In section 3, I will introduce the equilibrium concepts in the two auctions, propose and prove four propositions, and finally discuss the situation when changing GSP to the weighted version. Additional discussion will be made in section 4. In section 5 the conclusion will be drawn.

2 MODEL

In this paper we assume the sponsored search auction as an one-shot full-information game by using the following model

- A set of N advertisers (bidders). Each advertiser i has an private expected value-per-click v_i. Each advertiser would submit a bid b_i for a specific keyword. Let's denote the bid profile as b.
- A set of S possible positions (slots). Each position s has an estimated click-through-rate (CTR) x_s. We assume the positions have been ordered by CTR as $x_1 > x_2 > \ldots > x_S$. And we assume $S < N$ for analytical simplicity.
- The mechanism would allocate the positions as $d = \big(d\,(1)\,, d\,(2)\,, \ldots, d\,(S)\,\big)$ for the advertisers, where $d(s)$ is the advertiser allocated with the position s.
- The mechanism would require the advertiser allocated to with position s to pay p_s for clicking her ad for a certain period of time, not just for once.
- Thus we know, in a certain period of time, the profit (utility) for an advertiser i whose ad is allocated in position s is $v_i x_s - p_s$.

In the scenario of sponsored search auction, VCG and GSP are the two typical mechanisms which have attracted so much attentions.

GSP Auction. The first position (position with best CTR) is assigned to the advertiser with the highest bid, the second position to the advertiser with second-highest bid, and so on. For each position s, the advertiser $d(s)$ is required to pay the bid $b_{d(s+1)}$ for one click. Hence, the payment for a certain period of time is $p_s^{GSP} = x_s b_{d(s+1)}$ [8].

VCG Auction. The allocation rule for VCG is as same as that for GSP. For every position s, the advertiser $d(s)$ is required to pay the social cost. In this model, it is formulated as follows

$$\begin{aligned} p_s^{VCG} &= \sum_{j=1}^{s-1} x_j b_{d(j)} + \sum_{j=s+1}^{S} x_{j-1} b_{d(j)} - \sum_{j=1}^{s-1} x_j b_{d(j)} - \sum_{j=s+1}^{S} x_j b_{d(j)} \\ &= \sum_{j=s}^{S-1} (x_s - x_{s+1}) b_{d(s+1)}. \end{aligned} \tag{1}$$

Based on the above equation, we know there is a recursive formula as

$$p_s^{VCG} = (x_s - x_{s+1}) b_{d(s+1)} + p_{s+1}^{VCG} \qquad \forall s \in \{1, 2, \ldots, S-1\}, \tag{2}$$

where we have $p_S^{VCG} = 0$.

3 THEORETICAL ANALYSIS

In this section, we will first discuss some existing equilibrium concepts considered in the literatures for these two auctions. Then we propose the extension of VCG equilibrium with supporting propositions and the proofs. Finally, we investigate the situation in weighted version of GSP.

3.1 Equilibrium Concepts

As we all know in VCG auction, truth-telling is a unique dominant-strategy equilibrium. Other than the standard proof, under the model in this paper we can use another way to check it as follows.

Let's assume that the advertiser i is currently assigned the position s where the bid profile is b, and she wants to display her ad in the better position $s-1$ by changing her bid. Then the revenue will increase by $v_i(x_{s-1} - x_s)$, while according to formula (2), the payment will increase by $b_{d(s-1)}(x_{s-1} - x_s)$. Therefore, the marginal utility for advertiser i will be $(v_i - b_{d(s-1)})(x_{s-1} - x_s)$. According to the previous assumption of CTR, $(x_{s-1} - x_s)$ will always be positive. Thus getting the position $s-1$ from s by changing her bid for advertiser i is profitable if and only if $v_i > b_{d(s-1)}$. In the same way, getting the position $s+1$ from s is profitable if and only if $v_i < b_{d(s+1)}$. This implies that advertiser i maximizes her profit by submitting the bid as her true value-per-click, no matter what the other advertisers' bids are.

In GSP, we know there is no dominant-strategy equilibrium. According to the observation, however, advertiser slightly changing her bid without changing her position won't affect her profits. Thus it's reasonable to define Nash Equilibrium in GSP as a bid profile in which no advertiser has incentive to change to any other position. Formally, a bid profile b is a Nash equilibrium if $\forall\ i = d(s)$ we have

$$x_s(v_i - b_{d(s+1)}) \geq x_{s'}(v_i - b_{d(s')}) \quad \forall s' < s, \tag{3}$$

$$x_s(v_i - b_{d(s+1)}) \geq x_{s'}(v_i - b_{d(s'+1)}) \quad \forall s' > s, \tag{4}$$

which, you may have noticed, in an asymmetric form. Some subsets of the Nash equilibria with nice theoretical properties have been well analyzed, e.g., LEFNE by Edelman *et al.* [4] and SNE by Varian [5]. A bid profile b is LEFNE if it satisfies that any advertiser $i = d(s)$ can not improve her profit by changing her position to the position right above. It is also natural to define a refined version of LEFNE [9] as a bid profile also satisfying that any advertiser $i = d(s)$ can not improve her profit by changing her position to the position right below. Formally we say b is LEFNE if $\forall\ i = d(s)$ we have

$$x_s v_i - p_s^{GSP} \geq x_{s-1} v_i - p_{s-1}^{GSP}, \tag{5}$$

$$x_s v_i - p_s^{GSP} \geq x_{s+1} v_i - p_{s+1}^{GSP}. \tag{6}$$

A bid profile b is SNE if it satisfies that any advertiser $i = d(s)$ can not improve her profit by changing her position to any other position. Formally as

$$x_s(v_i - b_{d(s+1)}) \geq x_{s'}(v_i - b_{d(s')}) \quad \forall s' \in \{1, 2, \dots, S\}/\{s\}, \tag{7}$$

which is almost the same as the condition of Nash equilibrium. However, it is defined in a symmetric manner for simplicity while restricting the bid profiles to a subset of Nash equilibria. Varian has proved (Fact 5. in [5]) that SNE can also be characterized by (5) and (6), which means these two concepts are actually equivalent. We will refer to this concept as LEFNE for the rest of this paper.

3.2 Supporting Propositions

With a deeper investigation of the LEFNE concept in GSP, we can observe that it is not only valid in GSP. Actually it is also a reasonable equilibrium concept in any other mechanism where an advertiser slightly changing her bid without changing her position would not affect her utility. According to the VCG's payment rule, it is true in VCG. Therefore, it might be sensible to investigate the situation of applying LEFNE to VCG auction, which can be characterized by replacing p_s^{GSP} with p_s^{VCG} in (5) and (6). Here we propose and prove four propositions as follows.

Proposition 1. In VCG, the dominant-strategy equilibrium is also LEFNE.

Proof. Let's assume the bid profile b is the truth-telling dominant-strategy equilibrium. Given any position $s > 1$, we have

$$\left(x_s v_{d(s)} - p_s^{VCG}\right) - \left(x_{s-1} v_{d(s)} - p_{s-1}^{VCG}\right)$$

$$= (x_s - x_{s-1})v_{d(s)} - (p_s^{VCG} - p_{s-1}^{VCG}).$$

Substituting in the recursive formula (2), we have

$$(x_{s-1} - x_s)(b_{d(s)} - v_{d(s)}). \tag{8}$$

Since b is a truth-telling profile, we have $b_{d(s)} = v_{d(s)}$. The above equation is equal to zero, which satisfies condition (5). In the same way, we can prove that

$$\big(x_s v_{d(s)} - p_s^{VCG}\big) - \big(x_{s+1} v_{d(s)} - p_{s+1}^{VCG}\big) = (x_s - x_{s+1})(v_{d(s)} - b_{d(s+1)}). \tag{9}$$

Since we know $v_{d(s)} = b_{d(s)} \geq b_{d(s+1)}$ and $x_s > x_{s+1}$, the above equation is greater than or equal to zero, which satisfies condition (6). Hence, b is LEFNE. ∎

Proposition 1 tells us that LEFNE contains dominant-strategy equilibrium in VCG. And there is a set of LEFNEs. Hence LEFNE is an extension of dominant-strategy equilibrium.

Proposition 2. Under a LEFNE b in VCG, each advertiser i 's true value-per-click is less than or equal to her bid.

Proof. Since b is LEFNE, then (8) and (9) must be greater than or equal to zero so that b satisfies conditions (5) and (6), which implies that for any $s > 1$, we have $v_{d(s)} \leq b_{d(s)}$. ∎

Based on proposition 2, it's obvious that the revenue under all LEFNEs in VCG is greater or equal to that under the dominant-strategy equilibrium.

Proposition 3. In VCG, efficiency is achieved under all possible LEFNEs.

Proof. We can prove the contra-positivity of this proposition. Let's assume that a bid profile b is LEFNE in VCG. Suppose there exist two advertisers i and j with $v_i > v_j$ and $b_i \leq b_j$, which means the allocation is not efficient. Based on proposition 2, we have $v_j < v_i \leq b_i \leq b_j$. If advertiser j decrease her bid from b_j to $b'_j = v_j$ while resulting in changing her position from s to s' $(s < s')$, we can calculate the marginal utility as follows

$$\big(x_{s'} v_j - p_{s'}^{VCG}\big) - \big(x_s v_j - p_s^{VCG}\big)$$

$$= (x_{s'} - x_s)v_j + (p_s^{VCG} - p_{s'}^{VCG})$$

$$= -\Big(\sum_{t=s}^{s'-1} (x_t - x_{t+1})v_j - \sum_{t=s}^{s'-1} (p_t^{VCG} - p_{t+1}^{VCG})\Big)$$

$$= -\sum_{t=s}^{s'-1} (x_t - x_{t+1})(v_j - b_{d(t+1)}).$$

We know $v_j = b'_j = b_{d(s')}$. Thus $b_{d(t+1)} \geq b_{d(s')} = v_j \;\; \forall s \leq t < s' - 1$, and there exist at least one strict inequality $b_{d(t+1)} > v_j$ for a certain t because $b'_j < b_j$. Hence the above equation is strictly positive. Then changing from position s to s' by changing her bid from b_j to $b'_j = v_j$ is profitable for advertiser j, which conflicts with the assumption that b is LEFNE. ∎

Based on propositions 2 and 3, we can see that compared with the dominant-strategy equilibrium of VCG, the revenue under the LEFNE will increase while the efficiency is still guaranteed.

Proposition 4. The upper bound of the revenue among all the LEFNEs in VCG coincides with that in GSP.

Proof. First, let's calculate the upper bound of the revenues among all the LEFNEs in GSP. According to conditions (5) and (6), we have

$$(x_s - x_{s+1})v_{d(s)} + p_{s+1}^{GSP} \geq p_s^{GSP} \geq (x_s - x_{s+1})v_{d(s+1)} + p_{s+1}^{GSP}.$$

Because of $p_s^{GSP} = x_s b_{d(s+1)}$, we have

$$(x_{s-1} - x_s)v_{d(s-1)} + x_s b_{d(s+1)} \geq x_{s-1} b_{d(s)} \geq (x_{s-1} - x_s)v_{d(s)} + x_s b_{d(s+1)}. \quad (10)$$

Let's choose the upper bound in (10), then we have the recursive formula

$$x_{s-1} b_{d(s)} = x_s b_{d(s+1)} + (x_{s-1} - x_s)v_{d(s-1)}. \quad (11)$$

The solution of this recursive formula is

$$x_{s-1} b_{d(s)} = \sum_{t=s}^{S} v_{d(t-1)}(x_{t-1} - x_t) \;\; \forall s. \quad (12)$$

Thus the upper bound of the revenue among LEFNEs in GSP is

$$\begin{aligned} U^{GSP} &= \sum_{s=1}^{S} p_s^{GSP} = \sum_{s=1}^{S} x_s b_{d(s+1)} \\ &= \sum_{s=1}^{S} \sum_{t=s+1}^{S} v_{d(t-1)}(x_{t-1} - x_t) = \sum_{s=1}^{S-1} s v_{d(s)}(x_s - x_{s+1}). \end{aligned} \quad (13)$$

According to proposition 4 and the definition of LEFNE in VCG, if we consider the upper bound of the revenue among all LEFNEs in VCG, we have

$$b_{d(s)} = v_{d(s-1)} \;\; \forall s > 1,$$

and $b_{d(1)} = 2v_{d(1)}$. Then the upper bound of revenue for VCG is

$$U^{VCG} = \sum_{s=1}^{S} p_s^{VCG} = \sum_{s=1}^{S} \sum_{t=s+1}^{S} (x_{t-1} - x_t) b_{d(t)}$$
$$= \sum_{s=1}^{S} \sum_{t=s+1}^{S} (x_{t-1} - x_t) v_{d(t-1)} = \sum_{s=1}^{S-1} s v_{d(s)} (x_s - x_{s+1}). \quad (14)$$

Therefore, we have $U^{GSP} = U^{VCG}$. ∎

3.3 Weighted GSP

To make a successful sponsored search auction work, we have to consider the interests of all the three parties: users, advertisers and search engine companies. The advertisers want to show relevant ads so the user would be more likely to click on them. The users want to see more relevant ads with higher quality that can meet their real needs. A search engine's goal is to provide good experiences for both the advertiser and users, then they will come back and continue using the search service in the future. In this way, the role of a search engine here is more like a coordinator to provide a better matching between users and advertisers. Usually this consideration of the three parties and the relationship between them is called the ecosystem of the sponsored search auction. It is believed this consideration motivated the changing of GSP auction to a weighted version (wGSP) in recent years.

In wGSP, CTR is no longer just decided by the position and independent of the ad allocated to the position [8]. We assume that the CTR for an advertiser i in position s is a product of an advertiser-specific factor e_i ($0 < e_i \leq 1$) and a position-specific factor x_i, i.e., $e_i x_i$ rather than x_i. We usually consider e_s as a quality score of an ad. How to measure the quality score differs across the search engine companies. Usually the quality score has three main components: the click-through-rate for the ad, the relevancy with the keyword, and the quality of the landing page linked from the ad. For the allocation rule, wGSP will order the advertisers by $e_i b_i$. In this way, those advertisers with bad quality or not relevant will be dragged down even with high bids. For the payment rule, advertiser still pay the minimum amount to retain her position, which means $p_s^{wGSP} = (x_s e_{d(s)})(b_{d(s+1)} \frac{e_{d(s+1)}}{e_{d(s)}}) = x_s b_{d(s+1)} e_{d(s+1)}$. Under wGSP I calculate the upper bound of the revenue among all the LEFNEs (the calculation is provided in appendix A.) and the result is as follows

$$U^{wGSP} = \sum_{s=1}^{S-1} s v_{d(s)} e_{d(s)} (x_s - x_{s+1}). \quad (15)$$

Compared with (13), one may think $U^{wGSP} \leq U^{GSP}$ because $0 < e_{d(s)} \leq 1$. However, it's not true. The allocation rule for wGSP is different from that for

GSP and VCG so that the order of the S advertisers in (13) and (15) may be different. Then based on our analysis, we are not sure whether the upper bound of the revenue will definitely increase or decrease from GSP to wGSP. Therefore, different from proposition 4, we don't have a clear conclusion about the comparison between U^{wGSP} and U^{VCG}.

4 DISCUSSION

Let's get back to the two observations raised in the end of the first section, under the assumption that the LEFNE concept is valid in both VCG and GSP mechanisms. First, the use of LEFNE in VCG rather than only dominant-strategy equilibrium explains in some extent why the latter one is rarely reached, since it is only a specific equilibrium among all the LEFNEs in VCG. Second, based on propositions 1, 2, 4 and the proved theorem that the lower bound of revenue under LEFNE in GSP coincides with that under the truth-telling dominant-strategy equilibrium in VCG [5], we know that under LEFNEs the auctioneer's revenues in both GSP and VCG have the same range. This is consistent with some experiments reporting that GSP is not a clear winner in terms of revenue.

With the analysis for wGSP in the previous section, there is not a clear clue that the evolving from GSP to wGSP is driven by the revenue. However, it will increase the qualities of the ads shown to the users. Considering the long-term revenue, it is a reasonable improvement since it is consistent with the goal of a search engine company within the ecosystem of the ad auction. Actually this is reflected by the fact that wGSP has been employed by Google, Yahoo! and Microsoft Live since 2007 [10].

5 CONCLUSION

In this paper, we investigated the idea of extending the equilibrium concept in VCG from unique dominant-strategy equilibrium to the set of locally envy free Nash equilibrium introduced from GSP. We found this extension is valid in the sponsored search auction by the support of the following two main results. First, this extension of the equilibrium in VCG still guarantee the efficiency of the allocation. Second, considering LEFNE in both VCG and GSP mechanisms, not only the lower bound, but also the upper bound of the auctioneer's revenue coincide in both mechanisms. Hence the ranges of the revenue in both mechanism are same. These two results better explain the observations in some experiments. This extension of equilibrium concept might help to improve the prediction of the performance of VCG auction in the sponsored search auction.

References

[1] Hal R. Varian. The economics of internet search. *Angelo Costa lecture delivered in Rome*, February 2007.

[2] Krishna, Vijay and Perry, Motty. Efficient Mechanism Design. Game theory and information, EconWPA, May 1998.

[3] John Hegeman. Facebook's ad auction. *Talk at Ad Auctions Workshop*, May 2010.

[4] Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz. Internet advertising and the generalized second price auction: Selling billions of dollars worth of keywords. *American Economic Review*, 97, 2005.

[5] Hal R. Varian. Position auctions. *International Journal of Industrial Organization*, 25(6):1163–1178, December 2007.

[6] John H. Kagel, Ronald M Harstad, and Dan Levin. Information impact and allocation rules in auctions with affiliated private values: A laboratory study. *Econometrica*, 55(6):1275–1304, November 1987.

[7] John H. Kagel and D. Levin. Independent private value auctions: Bidder behavior in first, second and third-price auctions with varying numbers of bidders. Technical Report 13, 1988.

[8] Hal R. Varian. Online ad auctions. *American Economic Review*, 99(2):430–34, May 2009.

[9] Pennock D. Saberi A. Vohra R. Lahaie, S. Sponsored search auctions. In Tardos E. Vazirani V. (Eds.) Nisan N., Roughgarden T., editor, *Algorithmic Game Theory*, pages 699–716. Cambrige University Press, Cambridge, 2004.

[10] David Robert Martin Thompson and Kevin Leyton-Brown. Computational analysis of perfect-information position auctions. In *Proceedings of the 10th ACM conference on Electronic commerce*, EC '09, pages 51–60, 2009.

Appendix

A. Calculation of the upper bound of revenue in wGSP

To satisfy the conditions of LEFNE, for any advertiser $i = d(s)$ we have

$$e_i x_s v_i - p_s^{wGSP} \geq e_i x_{s-1} v_i - p_{s-1}^{wGSP}, \tag{16}$$

$$e_i x_s v_i - p_s^{wGSP} \geq e_i x_{s+1} v_i - p_{s+1}^{wGSP}. \tag{17}$$

Putting these two inequalities together we have

$$(x_s - x_{s+1}) v_{d(s)} e_{d(s)} + p_{s+1}^{wGSP} \geq p_s^{wGSP} \geq (x_s - x_{s+1}) v_{d(s+1)} e_{d(s+1)} + p_{s+1}^{wGSP}. \tag{18}$$

Recalling that $p_s^{wGSP} = x_s b_{d(s+1)} e_{d(s+1)}$, we can choose the upper bound in inequalities (18), then we have a recursive formula

$$x_{s-1} b_{d(s)} e_{d(s)} = x_s b_{d(s+1)} e_{d(s+1)} + (x_{s-1} - x_s) e_{d(s-1)} v_{d(s-1)}. \tag{19}$$

The solution to this formula is

$$x_{s-1} b_{d(s)} e_{d(s)} = \sum_{t=s}^{S} v_{d(t-1)} e_{d(t-1)} (x_{t-1} - x_t) \;\; \forall s. \tag{20}$$

Then we have

$$\begin{aligned} \bar{R}^{wGSP} &= \sum_{s=1}^{S} p_s^{wGSP} \\ &= \sum_{s=1}^{S} x_s b_{d(s+1)} e_{d(s+1)} \\ &= \sum_{s=1}^{S} \sum_{t=s+1}^{S} v_{d(t-1)} e_{d(t-1)} (x_{t-1} - x_t) \\ &= \sum_{s=1}^{S-1} s v_{d(s)} e_{d(s)} (x_s - x_{s+1}). \end{aligned} \tag{21}$$