Game Theory Week 3

Kevin Leyton-Brown

Lecture Overview

Domination

- What is strict domination?

Domination

- What is strict domination?
- What is very weak domination?

Domination

- What is strict domination?
- What is very weak domination?
- What is weak domination?

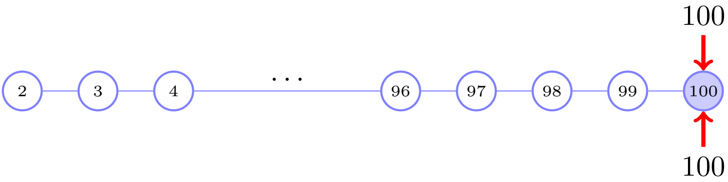
Domination

- What is strict domination?
- What is very weak domination?
- What is weak domination?
- How does iterated elimination of dominated strategies work?

Fun Game: Traveler's Dilemma

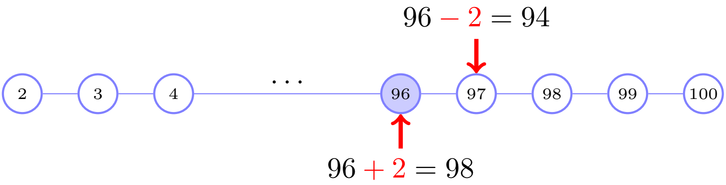
- Two players pick a number (2-100) simultaneously.

Fun Game: Traveler's Dilemma



- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.

Fun Game: Traveler's Dilemma

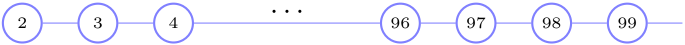


- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.

Fun Game: Traveler's Dilemma

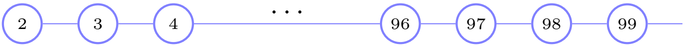
- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- Give this game a try. Play any opponent only once.

Fun Game: Traveler's Dilemma



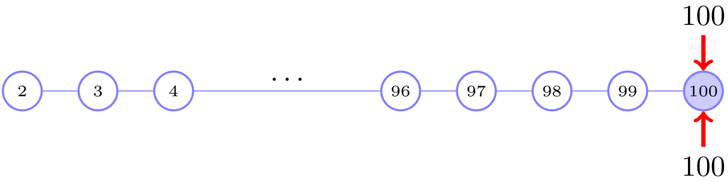
- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- Now play with bonus/penalty of 50.

Fun Game: Traveler's Dilemma



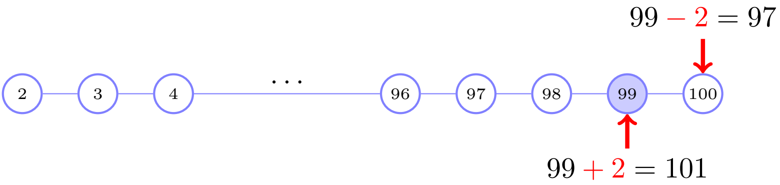
- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- What is the Nash equilibrium?

Fun Game: Traveler's Dilemma



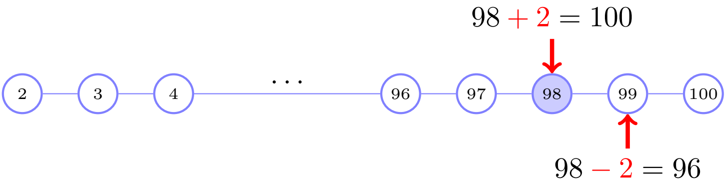
- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- Traveler's Dilemma has a unique Nash equilibrium.

Fun Game: Traveler's Dilemma



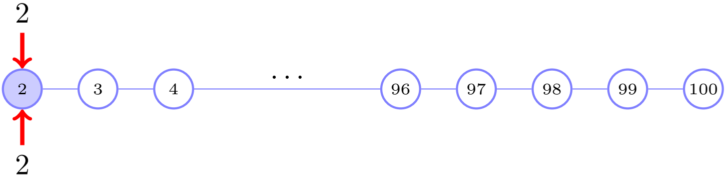
- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- Traveler's Dilemma has a unique Nash equilibrium.

Fun Game: Traveler's Dilemma



- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
    - Lower player gets lower number, plus bonus of 2.
    - Higher player gets lower number, minus penalty of 2.
- Traveler's Dilemma has a unique Nash equilibrium.

Fun Game: Traveler's Dilemma



- Two players pick a number (2-100) simultaneously.
- If they pick the same number, that is their payoff.
- If they pick different numbers:
  - Lower player gets lower number, plus bonus of 2.
  - Higher player gets lower number, minus penalty of 2.
- Traveler's Dilemma has a unique Nash equilibrium.

Domination

- If no pure strategy is dominated, can any mixed strategy be dominated? Why (not)?

Domination

- If no pure strategy is dominated, can any mixed strategy be dominated? Why (not)?
- If no pure strategy dominates another strategy, can any mixed strategy dominate another strategy? Why (not)?

Domination

- If no pure strategy is dominated, can any mixed strategy be dominated? Why (not)?
- If no pure strategy dominates another strategy, can any mixed strategy dominate another strategy? Why (not)?
- Does iterated removal preserve Nash equilibria? (All? Some?)
- Does the order of removal matter?

Lecture Overview

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rationalizable in matching pennies?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rationalizable in matching pennies?
  - is $cooperate$ rationalizable in prisoner's dilemma?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rationalizable in matching pennies?
  - is $cooperate$ rationalizable in prisoner's dilemma?
- Will there always exist a rationalizable strategy?

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rationalizable in matching pennies?
  - is $cooperate$ rationalizable in prisoner's dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.

Rationalizability

- Rather than ask what is irrational, ask what is a best response to some beliefs about the opponent
  - assumes opponent is rational
  - assumes opponent knows that you and the others are rational
  - ...
- Examples
  - is $heads$ rationalizable in matching pennies?
  - is $cooperate$ rationalizable in prisoner's dilemma?
- Will there always exist a rationalizable strategy?
  - Yes, equilibrium strategies are always rationalizable.
- Furthermore, in two-player games, rationalizable $\Leftrightarrow$ survives iterated removal of strictly dominated strategies.

Lecture Overview

Correlated Equilibrium

- What's the main idea here?

Formal definition

Definition (Correlated equilibrium)

Given an n-agent game $G = (N, A, u)$, a correlated equilibrium is a tuple (v, π, σ), where v is a tuple of random variables $v = (v_1, \ldots, v_n)$ with respective domains $D = (D_1, \ldots, D_n)$, π is a joint distribution over v, $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a vector of mappings $\sigma_i : D_i \mapsto A_i$, and for each agent i and every mapping $\sigma'_i : D_i \mapsto A_i$ it is the case that

$$\sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma_i(d_i), \ldots, \sigma_n(d_n)\right)$$
$$\geq \sum_{d \in D} \pi(d) u_i \left(\sigma_1(d_1), \ldots, \sigma'_i(d_i), \ldots, \sigma_n(d_n)\right).$$

Existence

Theorem

For every Nash equilibrium σ^* there exists a corresponding correlated equilibrium σ.

- This is easy to show:
  - let $D_i = A_i$
  - let $\pi(d) = \prod_{i \in N} \sigma_i^*(d_i)$
  - σ_i maps each d_i to the corresponding a_i.
- Thus, correlated equilibria always exist

Remarks

- Not every correlated equilibrium is equivalent to a Nash equilibrium
  - thus, correlated equilibrium is a weaker notion than Nash
- Any convex combination of the payoffs achievable under correlated equilibria is itself realizable under a correlated equilibrium
  - start with the Nash equilibria (each of which is a CE)
  - introduce a second randomizing device that selects which CE the agents will play
  - regardless of the probabilities, no agent has incentive to deviate
  - the probabilities can be adjusted to achieve any convex combination of the equilibrium payoffs
  - the randomizing devices can be combined

Lecture Overview

1. Domination
2. Rationalizability
3. Correlated Equilibrium
4. Computing CE
5. Computational problems in domination

Computing CE

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i \in a} p(a) u_i(a'_i, a_{-i}) \quad \forall i \in N, \forall a_i, a'_i \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$

Computing CE

$$\sum_{a\in A|a_i\in a} p(a)u_i(a) \geq \sum_{a\in A|a_i\in a} p(a)u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a\in A} p(a) = 1$$

- variables: $p(a)$; constants: $u_i(a)$
- we could find the social-welfare maximizing CE by adding an objective function

$$\text{maximize:} \quad \sum_{a\in A} p(a) \sum_{i\in N} u_i(a).$$

Why are CE easier to compute than NE?

$$\sum_{a \in A | a_i \in a} p(a) u_i(a) \geq \sum_{a \in A | a_i' \in a} p(a) u_i(a_i', a_{-i}) \quad \forall i \in N, \forall a_i, a_i' \in A_i$$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

- intuitively, correlated equilibrium has only a single randomization over outcomes, whereas in NE this is constructed as a product of independent probabilities.
- To change this program so that it finds NE, the first constraint would be

$$\sum_{a \in A} u_i(a) \prod_{j \in N} p_j(a_j) \geq \sum_{a \in A} u_i(a_i', a_{-i}) \prod_{j \in N \setminus \{i\}} p_j(a_j) \quad \forall i \in N, \forall a_i' \in A_i.$$

- This is a nonlinear constraint!

Lecture Overview

1. Domination
2. Rationalizability
3. Correlated Equilibrium
4. Computing CE
5. Computational problems in domination

Computational Problems in Domination

- Identifying strategies dominated by a pure strategy
- Identifying strategies dominated by a mixed strategy
- Identifying strategies that survive iterated elimination
- Asking whether a strategy survives iterated elimination under all elimination orderings
- We'll assume that i's utility function is strictly positive everywhere (why is this OK?)

Is s_i strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

```
for all pure strategies a_i ∈ A_i for player i where a_i ≠ s_i do
    dom ← true
    for all pure strategy profiles a_{-i} ∈ A_{-i} for the players other than i
    do
        if u_i(s_i, a_{-i}) ≥ u_i(a_i, a_{-i}) then
            dom ← false
            break
        end if
    end for
    if dom = true then return true
end for
return false
```

Is s_i strictly dominated by any pure strategy?

Try to identify some pure strategy that is strictly better than s_i for any pure strategy profile of the others.

for all pure strategies $a_i \in A_i$ for player i where $a_i \neq s_i$ **do**
  $dom \leftarrow true$
  for all pure strategy profiles $a_{-i} \in A_{-i}$ for the players other than i **do**
    if $u_i(s_i, a_{-i}) \geq u_i(a_i, a_{-i})$ **then**
      $dom \leftarrow false$
      break
    end if
  end for
  if $dom = true$ **then return** $true$
end for
return $false$

- What is the complexity of this procedure?
- Why don't we have to check mixed strategies of $-i$?
- Minor changes needed to test for weak, very weak dominance.

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?

Constraints for determining whether s_i is strictly dominated by any mixed strategy

$$\sum_{j \in A_i} p_j u_i(a_j, a_{-i}) > u_i(s_i, a_{-i}) \qquad \forall a_{-i} \in A_{-i}$$

$$p_j \geq 0 \qquad \forall j \in A_i$$

$$\sum_{j \in A_i} p_j = 1$$

- What's wrong with this program?
  - strict inequality in first constraint: we don't have an LP

LP for determining whether s_i is strictly dominated by any mixed strategy

$$
\begin{aligned}
\text{minimize} \quad & \sum_{j \in A_i} p_j \\
\text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& p_j \geq 0 && \forall j \in A_i
\end{aligned}
$$

- This is clearly an LP. Why is it a solution to our problem?

LP for determining whether s_i is strictly dominated by any mixed strategy

$$\begin{aligned} \text{minimize} \quad & \sum_{j \in A_i} p_j \\ \text{subject to} \quad & \sum_{j \in A_i} p_j u_i(a_j, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\ & p_j \geq 0 && \forall j \in A_i \end{aligned}$$

- This is clearly an LP. Why is it a solution to our problem?
  - if a solution exists with $\sum_j p_j < 1$ then we can add $1 - \sum_j p_j$ to some p_k and we'll have a dominating mixed strategy (since utility was assumed to be positive everywhere)
- Our original approach works for very weak domination
- For weak domination we can use that program with a different objective function trick.

Identifying strategies that survive iterated elimination

- This can be done by repeatedly solving our LPs: solving a polynomial number of LPs is still in $\mathcal{P}$.
  - Checking whether every pure strategy of every player is dominated by any other mixed strategy requires us to solve at worst $\sum_{i \in N} |A_i|$ linear programs.
  - Each step removes one pure strategy for one player, so there can be at most $\sum_{i \in N}(|A_i| - 1)$ steps.
  - Thus we need to solve $O((n \cdot \max_i |A_i|)^2)$ linear programs.

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

Further questions about iterated elimination

1. **(Strategy Elimination)** Does there exist some elimination path under which the strategy s_i is eliminated?
2. **(Reduction Identity)** Given action subsets $A'_i \subseteq A_i$ for each player i, does there exist a maximally reduced game where each player i has the actions A'_i?
3. **(Uniqueness)** Does every elimination path lead to the same reduced game?
4. **(Reduction Size)** Given constants k_i for each player i, does there exist a maximally reduced game where each player i has exactly k_i actions?

- For iterated strict dominance these problems are all in $\mathcal{P}$.
- For iterated weak or very weak dominance these problems are all $\mathcal{NP}$-complete.