# From Optimality to Equilibrium

Lecture 4

From Optimality to Equilibrium

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# Lecture Overview



## 2 Pareto Optimality

## 3 Best Response and Nash Equilibrium

## 4 Mixed Strategies

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# Non-Cooperative Game Theory

• What is it?

Recap

• mathematical study of interaction between rational, self-interested agents

- Why is it called non-cooperative?
  - while it's most interested in situations where agents' interests conflict, it's not restricted to these settings
  - the key is that the individual is the basic modeling unit, and that individuals pursue their own interests
    - cooperative/coalitional game theory has teams as the central unit, rather than agents

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 Recap
 Pareto Optimality
 Best Response and Nash Equilibrium
 Mixed Strategies

 Defining Games
 Image: Compared Strategies
 Image: Compared Strategies
 Image: Compared Strategies

- Finite, *n*-person game:  $\langle N, A, u \rangle$ :
  - N is a finite set of n players, indexed by i
  - $A = A_1 \times \ldots \times A_n$ , where  $A_i$  is the action set for player i
    - $a \in A$  is an action profile, and so A is the space of action profiles
  - $u = \langle u_1, \dots, u_n \rangle$ , a utility function for each player, where  $u_i : A \mapsto \mathbb{R}$
- Writing a 2-player game as a matrix:
  - row player is player 1, column player is player 2
  - rows are actions  $a \in A_1$ , columns are  $a' \in A_2$
  - cells are outcomes, written as a tuple of utility values for each player

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## Prisoner's dilemma

#### Prisoner's dilemma is any game

 $\begin{array}{c|c} C & D \\ \hline \\ C & a, a & b, c \\ \hline \\ D & c, b & d, d \end{array}$ 

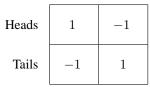
with c > a > d > b.

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# Games of Pure Competition

Players have exactly opposed interests

- There must be precisely two players (otherwise they can't have exactly opposed interests)
- For all action profiles  $a \in A$ ,  $u_1(a) + u_2(a) = c$  for some constant c
  - Special case: zero sum



# Heads Tails

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# Games of Cooperation

Players have exactly the same interests.

• no conflict: all players want the same things

• 
$$\forall a \in A, \forall i, j, u_i(a) = u_j(a)$$

Left	1	0
Right	0	1

Left

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# General Games: Battle of the Sexes

The most interesting games combine elements of cooperation *and* competition.

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- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?

Analyzing Games

- We've defined some canonical games, and thought about how to play them. Now let's examine the games from the outside
- From the point of view of an outside observer, can some outcomes of a game be said to be better than others?
  - we have no way of saying that one agent's interests are more important than another's
  - intuition: imagine trying to find the revenue-maximizing outcome when you don't know what currency has been used to express each agent's payoff
- Are there situations where we can still prefer one outcome to another?

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'
  - $\bullet\,$  in this case, it seems reasonable to say that o is better than o'
  - we say that *o* Pareto-dominates *o*'.

- Idea: sometimes, one outcome o is at least as good for every agent as another outcome  $o^\prime$ , and there is some agent who strictly prefers o to  $o^\prime$ 
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• An outcome  $o^*$  is Pareto-optimal if there is no other outcome that Pareto-dominates it.

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Pareto Optimality

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- An outcome o<sup>\*</sup> is Pareto-optimal if there is no other outcome that Pareto-dominates it.
  - can a game have more than one Pareto-optimal outcome?
  - does every game have at least one Pareto-optimal outcome?

## Pareto Optimal Outcomes in Example Games

$$C$$
  $D$ 

$$\begin{array}{c|ccc} C & -1, -1 & -4, 0 \\ \hline D & 0, -4 & -3, -3 \end{array}$$

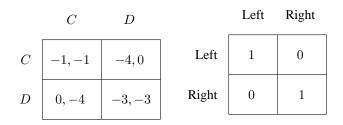
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## Pareto Optimal Outcomes in Example Games



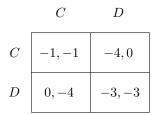
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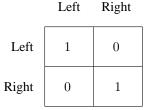
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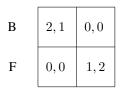
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## Pareto Optimal Outcomes in Example Games





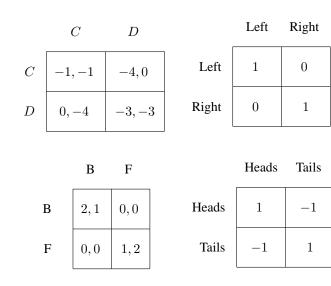
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## Pareto Optimal Outcomes in Example Games



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# Lecture Overview





### 3 Best Response and Nash Equilibrium

### 4 Mixed Strategies

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• If you knew what everyone else was going to do, it would be easy to pick your own action

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• If you knew what everyone else was going to do, it would be easy to pick your own action

• Let 
$$a_{-i} = \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n \rangle$$
.

• now 
$$a = (a_{-i}, a_i)$$

• Best response:  $a_i^* \in BR(a_{-i})$  iff  $\forall a_i \in A_i, u_i(a_i^*, a_{-i}) \ge u_i(a_i, a_{-i})$ 

- Now let's return to the setting where no agent knows anything about what the others will do
- What can we say about which actions will occur?

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- What can we say about which actions will occur?

- Idea: look for stable action profiles.
- $a = \langle a_1, \ldots, a_n \rangle$  is a ("pure strategy") Nash equilibrium iff  $\forall i, a_i \in BR(a_{-i})$ .

# Nash Equilibria of Example Games

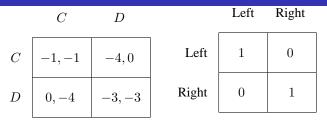
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# Nash Equilibria of Example Games



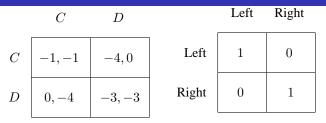
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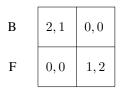
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# Nash Equilibria of Example Games



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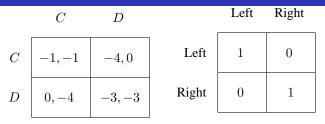
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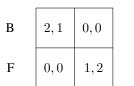
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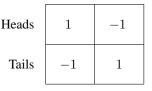
# Nash Equilibria of Example Games



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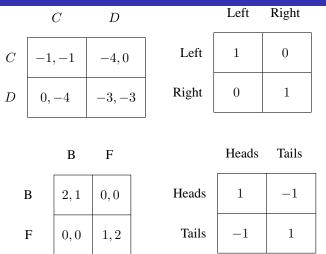


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# Nash Equilibria of Example Games



The paradox of Prisoner's dilemma: the Nash equilibrium is the only non-Pareto-optimal outcome!

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# Lecture Overview



2 Pareto Optimality

3 Best Response and Nash Equilibrium

## Mixed Strategies

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Mixed Strategies

- It would be a pretty bad idea to play any deterministic strategy in matching pennies
- Idea: confuse the opponent by playing randomly
- Define a strategy  $s_i$  for agent i as any probability distribution over the actions  $A_i$ .
  - pure strategy: only one action is played with positive probability
  - mixed strategy: more than one action is played with positive probability
    - these actions are called the support of the mixed strategy
- Let the set of all strategies for i be  $S_i$
- Let the set of all strategy profiles be  $S = S_1 \times \ldots \times S_n$ .

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## Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile  $s \in S$ ?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell

# Utility under Mixed Strategies

- What is your payoff if all the players follow mixed strategy profile s ∈ S?
  - We can't just read this number from the game matrix anymore: we won't always end up in the same cell
- Instead, use the idea of expected utility from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$
$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

# Best Response and Nash Equilibrium

Our definitions of best response and Nash equilibrium generalize from actions to strategies.

- Best response:
  - $s_i^* \in BR(s_{-i})$  iff  $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \ge u_i(s_i, s_{-i})$

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# Best Response and Nash Equilibrium

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- Nash equilibrium:
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# Best Response and Nash Equilibrium

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- Best response:
  - $s_i^* \in BR(s_{-i})$  iff  $\forall s_i \in S_i, u_i(s_i^*, s_{-i}) \ge u_i(s_i, s_{-i})$
- Nash equilibrium:
  - $s = \langle s_1, \ldots, s_n \rangle$  is a Nash equilibrium iff  $\forall i, s_i \in BR(s_{-i})$
- Every finite game has a Nash equilibrium! [Nash, 1950]
  e.g., matching pennies: both players play heads/tails 50%/50%

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## Computing Mixed Nash Equilibria: Battle of the Sexes

	В	F
В	2, 1	0, 0
F	0,0	1, 2

- It's hard in general to compute Nash equilibria, but it's easy when you can guess the support
- For BoS, let's look for an equilibrium where all actions are part of the support

## Computing Mixed Nash Equilibria: Battle of the Sexes



- Let player 2 play B with p, F with 1 p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

## Computing Mixed Nash Equilibria: Battle of the Sexes



- Let player 2 play B with p, F with 1 p.
- If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B (why?)

$$u_1(B) = u_1(F)$$
  
 $2p + 0(1-p) = 0p + 1(1-p)$   
 $p = \frac{1}{3}$ 

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# Computing Mixed Nash Equilibria: Battle of the Sexes



- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?

## Computing Mixed Nash Equilibria: Battle of the Sexes



- Likewise, player 1 must randomize to make player 2 indifferent.
  - Why is player 1 willing to randomize?
- Let player 1 play B with q, F with 1-q.

$$u_2(B) = u_2(F)$$

$$q + 0(1 - q) = 0q + 2(1 - q)$$

$$q = \frac{2}{3}$$
Thus the mixed strategies  $(\frac{2}{3}, \frac{1}{3})$ ,  $(\frac{1}{3}, \frac{2}{3})$  are a Nash equilibrium.

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# Interpreting Mixed Strategy Equilibria

What does it mean to play a mixed strategy? Different interpretations:

- Randomize to confuse your opponent
  - consider the matching pennies example
- Players randomize when they are uncertain about the other's action
  - consider battle of the sexes
- Mixed strategies are a concise description of what might happen in repeated play: count of pure strategies in the limit
- Mixed strategies describe population dynamics: 2 agents chosen from a population, all having deterministic strategies. MS is the probability of getting an agent who will play one PS or another.

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