Reasoning Under Uncertainty: Variable Elimination

CPSC 322 – Uncertainty 7

Textbook §10.5

Lecture Overview

Factors and Assigning Variables

- A factor is a representation of a function from a tuple of random variables into a number.
- A factor denotes a distribution over the given tuple of variables in some (unspecified) context

- We can make new factors out of an existing factor
- Our first operation: assign some or all of the variables of a factor.
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.
- The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1, \ldots, X_j = v_j}$

Summing out variables

Our second operation: we can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$(\sum_{X_1} f)(X_2, \ldots, X_j) = f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)$$

Multiplying factors

- Our third operation: factors can be multiplied together.
- The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) = f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Lecture Overview

1. Recap
2. Variable Elimination
3. Variable Elimination Example

Probability of a conjunction

- Suppose the variables of the belief network are $X_1, \ldots, X_n$.
- What we want to compute: the factor $P(X_q, X_{o_1} = v_1, \ldots, X_{o_j} = v_j)$
- We can compute $P(X_q, X_{o_1} = v_1, \ldots, X_{o_j} = v_j)$ by summing out the variables $X_{s_1}, \ldots, X_{s_k} = \{X_1, \ldots, X_n\} \setminus \{X_q, X_{o_1}, \ldots, X_{o_j}\}$.
- We sum out these variables one at a time
  - the order in which we do this is called our elimination ordering.

$$P(X_q, X_{o_1} = v_1, \ldots, X_{o_j} = v_j)$$
$$= \sum_{X_{s_k}} \cdots \sum_{X_{s_1}} P(X_1, \ldots, X_n)_{X_{o_1} = v_1, \ldots, X_{o_j} = v_j}.$$

Probability of a conjunction

- What we know: the factors $P(X_i|pX_i)$.
- Using the chain rule and the definition of a belief network, we can write $P(X_1, \ldots, X_n)$ as $\prod_{i=1}^{n} P(X_i|pX_i)$. Thus:

$$
\begin{aligned}
& P(X_q, X_{o_1} = v_1, \ldots, X_{o_j} = v_j) \\
& \quad = \sum_{X_{s_k}} \cdots \sum_{X_{s_1}} P(X_1, \ldots, X_n)_{X_{o_1} = v_1, \ldots, X_{o_j} = v_j} . \\
& \quad = \sum_{X_{s_k}} \cdots \sum_{X_{s_1}} \prod_{i=1}^{n} P(X_i|pX_i)_{X_{o_1} = v_1, \ldots, X_{o_j} = v_j} .
\end{aligned}
$$

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?

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- How can we compute $\sum_{X_{s_1}} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?

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  - this takes only 7 multiplications or additions
- How can we compute $\sum_{X_{s_1}} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?
- Factor out those terms that don't involve X_{s_1}:

$$\left(\underbrace{\prod_{i|X_{s_1} \notin \{X_i\} \cup pX_i} P(X_i|pX_i)}_{\text{(terms that do not involve } X_{s_i}\text{)}} \right) \left(\sum_{X_{s_1}} \underbrace{\prod_{i|X_{s_1} \in \{X_i\} \cup pX_i} P(X_i|pX_i)}_{\text{(terms that involve } X_{s_i}\text{)}} \right)$$

Summing out a variable efficiently

To sum out a variable X_{s_j} from a product $f_1, \ldots, f_k$ of factors:

- Partition the factors into
  - those that don't contain X_{s_j}, say $f_1, \ldots, f_i$,
  - those that contain X_{s_j}, say $f_{i+1}, \ldots, f_k$

We know:

$$\sum_{X_{s_j}} f_1 \times \cdots \times f_k = (f_1 \times \cdots \times f_i) \left(\sum_{X_{s_j}} f_{i+1} \times \cdots \times f_k \right).$$

- $\left(\sum_{X_{s_j}} f_{i+1} \times \cdots \times f_k \right)$ is a new factor; let's call it f'.
- Now we have $\sum_{X_{s_j}} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f'$.
- Store f' explicitly, and discard $f_{i+1}, \ldots, f_k$.
  - Now we've summed out X_{s_j}.

Variable elimination algorithm

To compute $P(X_q|X_{o_1} = v_1 \wedge \ldots \wedge X_{o_j} = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- For each of the other variables $X_{s_i} \in \{X_{s_1}, \ldots, X_{s_k}\}$, sum out X_{s_i}
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(X_q)$ by $\sum_{X_q} f(X_q)$.

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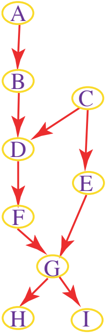
Variable elimination example

Compute $P(G|H = h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A, B, C, D, E, F, G, H, I)$
- $P(G, H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B, C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F, E) \cdot P(H|G) \cdot P(I|G)$



Variable elimination example

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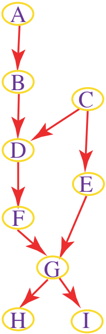
- $P(G,H) = \sum_{A,B,C,D,E,F,I} P(A) \cdot P(B|A) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate A: $P(G,H) = \sum_{B,C,D,E,F,I} f_1(B) \cdot P(C) \cdot P(D|B,C) \cdot P(E|C) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$



- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$

Variable elimination example

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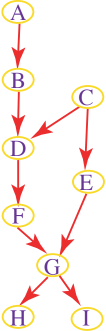
- Eliminate C: $P(G,H) = \sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$

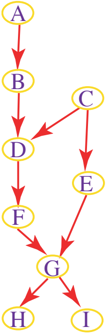


- $f_1(B) := \sum_{a \in dom(A)} P(A=a) \cdot P(B|A=a)$
- $f_2(B,D,E) := \sum_{c \in dom(C)} P(C=c) \cdot P(D|B,C=c) \cdot P(E|C=c)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) =$
  $\sum_{B,D,E,F,I} f_1(B) \cdot f_2(B,D,E) \cdot P(F|D) \cdot P(G|F,E) \cdot P(H|G) \cdot P(I|G)$
- Eliminate E:
  $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$



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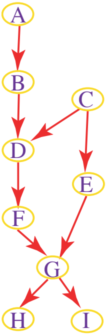
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G,H) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot P(H|G) \cdot P(I|G)$
- Observe $H=h_1$:

  $P(G,H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$



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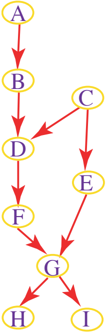
Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F,I} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot P(I|G)$
- Eliminate I:

  $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$



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- $f_4(G) := P(H=h_1|G)$
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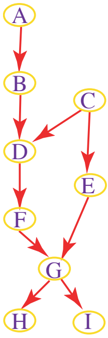
Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{B,D,F} f_1(B) \cdot f_3(B,D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate B:
  $P(G, H=h_1) = \sum_{D,F} f_6(D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$



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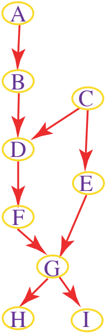
- $f_3(B,D,F,G) := \sum_{e \in dom(E)} f_2(B,D,E=e) \cdot P(G|F,E=e)$
- $f_4(G) := P(H=h_1|G)$
- $f_5(G) := \sum_{i \in dom(I)} P(I=i|G)$
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_{D,F} f_6(D,F,G) \cdot P(F|D) \cdot f_4(G) \cdot f_5(G)$
- Eliminate D: $P(G, H=h_1) = \sum_{F} f_7(F,G) \cdot f_4(G) \cdot f_5(G)$



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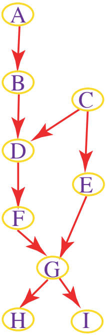
- $f_6(D,F,G) := \sum_{b \in dom(B)} f_1(B=b) \cdot f_3(B=b,D,F,G)$
- $f_7(F,G) := \sum_{d \in dom(D)} f_6(D=d,F,G) \cdot P(F|D=d)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = \sum_F f_7(F,G) \cdot f_4(G) \cdot f_5(G)$
- Eliminate F: $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$



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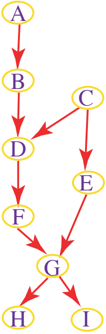
- $f_8(G) := \sum_{f \in dom(F)} f_7(F=f,G)$

Variable elimination example

Compute $P(G|H=h_1)$. Elimination order: A, C, E, H, I, B, D, F

- $P(G, H=h_1) = f_8(G) \cdot f_4(G) \cdot f_5(G)$
- Normalize: $P(G|H=h_1) = \frac{P(G,H=h_1)}{\sum_{g \in dom(G)} P(G,H=h_1)}$



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- And... why did we bother learning conditional independence? Does it help us at all?
  - yes—we use the chain rule decomposition right at the beginning

- Can we use our knowledge of conditional independence to make this calculation even simpler?
  - yes—there are some variables that we don't have to sum out
  - intuitively, they're the ones that are "pre-summed-out" in our tables
  - example: summing out I on the previous slide

One Last Trick

One last trick to simplify calculations: we can repeatedly eliminate all leaf nodes that are neither observed nor queried, until we reach a fixed point.

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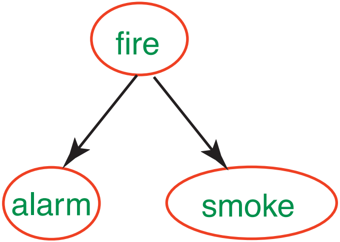

Can we justify that on a three-node graph—Fire, Alarm, and Smoke—when we ask for:

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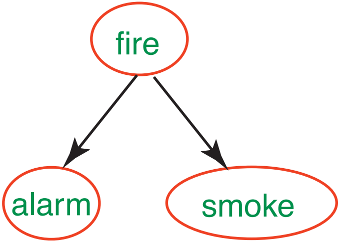

Can we justify that on a three-node graph—Fire, Alarm, and Smoke—when we ask for:

- $P(Fire)$?
- $P(Fire \mid Alarm)$?