Reasoning Under Uncertainty: Belief Networks

CPSC 322 – Uncertainty 4

Textbook §6.3

Lecture Overview

1. Recap
2. Belief Networks

Marginal independence

Definition (marginal independence)

Random variable X is marginally independent of random variable Y if, for all $x_i \in dom(X)$, $y_j \in dom(Y)$ and $y_k \in dom(Y)$,

$$\begin{aligned} P(X = x_i | Y = y_j) \\ &= P(X = x_i | Y = y_k) \\ &= P(X = x_i). \end{aligned}$$

That is, knowledge of Y's value doesn't affect your belief in the value of X.

Conditional Independence

- Sometimes, two random variables might not be marginally independent. However, they can *become* independent after we observe some third variable.

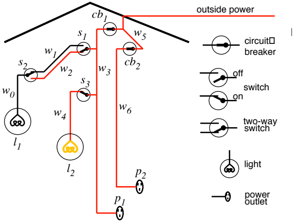
Definition

Random variable X is conditionally independent of random variable Y given random variable Z if, for all $x_i \in dom(X)$, $y_j \in dom(Y)$, $y_k \in dom(Y)$ and $z_m \in dom(Z)$,

$$\begin{aligned} P(X = x_i | Y = y_j \wedge Z = z_m) \\ = \quad P(X = x_i | Y = y_k \wedge Z = z_m) \\ = \quad P(X = x_i | Z = z_m). \end{aligned}$$

- That is, knowledge of Y's value doesn't affect your belief in the value of X, given a value of Z.

More examples of conditional independence



- Whether light $l1$ is lit is independent of the position of light switch $s2$ given whether there is power in wire w_0.
  - two random variables that are not marginally independent can still be conditionally independent
- Every other variable may be independent of whether light $l1$ is lit given whether there is power in wire w_0 and the status of light $l1$ (if it's ok, or if not, how it's broken).

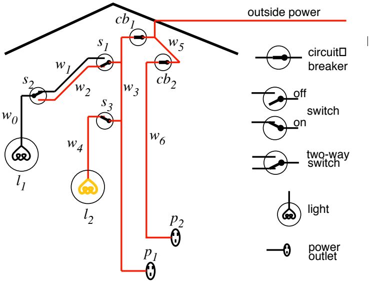
More examples of conditional independence

- The probability that the Canucks will win the Stanley Cup is independent of whether light $l1$ is lit given whether there is outside power.
  - sometimes, when two random variables are marginally independent, they're also conditionally independent given a third variable.
- But not always...
  - Let C_1 be the proposition that coin 1 is heads; let C_2 be the proposition that coin 2 is heads; let B be the proposition that coin 1 and coin 2 are both either heads or tails.
  - $P(C_1|C_2) = P(C_1)$: C_1 and C_2 are marginally independent.
  - But $P(C_1|C_2, B) \neq P(C_1|B)$: if I know both C_2 and B, I know C_1 exactly, but if I only know B I know nothing.
  - Hence C_1 and C_2 are *not* conditionally independent given B.

Lecture Overview

Idea of belief networks

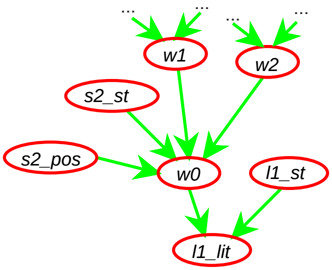
Whether $l1$ is lit ($L1\_lit$) depends only on the status of the light ($L1\_st$) and whether there is power in wire $w0$. Thus, $L1\_lit$ is independent of the other variables given $L1\_st$ and $W0$. In a belief network, $W0$ and $L1\_st$ are parents of $L1\_lit$.



Similarly, $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Idea of belief networks

Whether $l1$ is lit ($L1\_lit$) depends only on the status of the light ($L1\_st$) and whether there is power in wire $w0$. Thus, $L1\_lit$ is independent of the other variables given $L1\_st$ and $W0$. In a belief network, $W0$ and $L1\_st$ are parents of $L1\_lit$.



Similarly, $W0$ depends only on whether there is power in $w1$, whether there is power in $w2$, the position of switch $s2$ ($S2\_pos$), and the status of switch $s2$ ($S2\_st$).

Components of a belief network

Definition (belief network)

A belief network consists of:

- a directed acyclic graph with nodes labeled with random variables
- a domain for each random variable
- a set of conditional probability tables for each variable given its parents (which includes prior probabilities for nodes with no parents).

Relating BNs to the joint

Belief networks are compact representations of the joint.

To encode the joint as a BN:

1. Totally order the variables of interest: $X_1, \ldots, X_n$
2. Write down the chain rule decomposition of the joint, using this ordering: $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | X_{i-1}, \ldots, X_1)$
3. For every variable X_i, find the smallest set $pX_i \subseteq \{X_1, \ldots, X_{i-1}\}$ such that $P(X_i | X_{i-1}, \ldots, X_1) = P(X_i | pX_i)$.
   - If $pX_i \neq \{X_1, \ldots, X_{i-1}\}$, X_i is conditionally independent of some of its ancestors given pX_i.
4. Now we can write $P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i | pX_i)$
5. Construct the BN:
   - Nodes are variables
   - Incoming edges to each variable X_i from each variable in pX_i
   - Conditional probability table for variable X_i: $P(X_i | pX_i)$