Reasoning Under Uncertainty: Conditional Probability

CPSC 322 - Uncertainty 2

Textbook §6.1

Reasoning Under Uncertainty: Conditional Probability

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- Probability Distributions
- 3 Conditional Probability
- 4 Bayes' Theorem

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Possible World Semantics

Probability is a formal measure of uncertainty.

- A random variable is a variable that is randomly assigned one of a number of different values.
- The domain of a variable X, written dom(X), is the set of values X can take.
- A possible world specifies an assignment of one value to each random variable.
- $w \models X = x$ means variable X is assigned value x in world w.
- Let Ω be the set of all possible worlds.
- Define a nonnegative measure $\mu(w)$ to each world w so that the measures of the possible worlds sum to 1.
- The probability of proposition f is defined by:

$$P(f) = \sum_{w \models f} \mu(w).$$



Probability Distributions

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Probability Distributions

Definition (probability distribution)

A probability distribution P on a random variable X is a function $dom(X) \to [0,1]$ such that

$$x \mapsto P(X = x).$$

• When dom(X) is infinite we need a probability density function.

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Joint Distribution

When there are multiple random variables, their joint distribution is a probability distribution over the variables' Cartesian product

- E.g., P(X, Y, Z) means $P(\langle X, Y, Z \rangle)$.
- Think of a joint distribution over *n* variables as an *n*-dimensional table
- Each entry, indexed by $X_1 = x_1, \ldots, X_n = x_n$, corresponds to $P(X_1 = x_1 \land \ldots \land X_n = x_n)$.
- The sum of entries across the whole table is 1.

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Joint Distribution Example

Consider the following example, describing what a given day might be like in Vancouver.

- we have two random variables:
 - *weather*, with domain {Sunny, Cloudy};
 - *temperature*, with domain {Hot, Mild, Cold}.
- Then we have the joint distribution

P(weather, temperature) given as follows:

		temperature		
		Hot	Mild	Cold
weather	Sunny	0.10	0.20	0.10
weather	Cloudy	0.05	0.35	0.20

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Marginalization

Given the joint distribution, we can compute distributions over smaller sets of variables through marginalization:

- E.g., $P(X,Y) = \sum_{z \in dom(Z)} P(X,Y,Z=z).$
- This corresponds to summing out a dimension in the table.
- The new table still sums to 1.

Marginalization Example

		temperature		
		Hot	Mild	Cold
weather	Sunny	0.10	0.20	0.10
weumer	Cloudy	0.05	0.35	0.20

If we marginalize out weather, we get

$$P(temperature) = \begin{array}{|c|c|} Hot & Mild & Cold \\ \hline 0.15 & 0.55 & 0.30 \\ \hline \end{array}$$

If we marginalize out *temperature*, we get

$$P(weather) = \begin{array}{|c|c|} Sunny & Cloudy \\ \hline 0.40 & 0.60 \end{array}$$



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Conditioning

- Probabilistic conditioning specifies how to revise beliefs based on new information.
- You build a probabilistic model taking all background information into account. This gives the prior probability.
- All other information must be conditioned on.
- If evidence e is all of the information obtained subsequently, the conditional probability P(h|e) of h given e is the posterior probability of h.

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Semantics of Conditional Probability

- Evidence e rules out possible worlds incompatible with e.
- We can represent this using a new measure, $\mu_e,$ over possible worlds

$$\mu_e(\omega) = \begin{cases} \frac{1}{P(e)} \times \mu(\omega) & \text{if } \omega \models e \\ 0 & \text{if } \omega \not\models e \end{cases}$$

Definition

The conditional probability of formula h given evidence e is

$$P(h|e) = \sum_{\substack{\omega \models h}} \mu_e(w)$$
$$= \frac{P(h \land e)}{P(e)}$$

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Conditional Probability Example

		temperature		
		Hot	Mild	Cold
weather	Sunny	0.10	0.20	0.10
weuther	Cloudy	y 0.05 0.35	0.20	

If we condition on weather = Sunny, we get

$$P(temperature|Weather = Sunny) = \begin{array}{|c|c|c|} Hot & Mild & Cold \\ \hline 0.25 & 0.50 & 0.25 \end{array}$$

Conditioning on temperature, we get P(weather|temperature):

weather Sunny Cloudy Cl

Note that each column now sums to one.

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Chain Rule

Definition (Chain Rule)

$$P(f_1 \wedge f_2 \wedge \dots \wedge f_n)$$

$$= P(f_n | f_1 \wedge \dots \wedge f_{n-1}) \times P(f_1 \wedge \dots \wedge f_{n-1})$$

$$= P(f_n | f_1 \wedge \dots \wedge f_{n-1}) \times P(f_{n-1} | f_1 \wedge \dots \wedge f_{n-2}) \times P(f_1 \wedge \dots \wedge f_{n-2})$$

$$= P(f_n | f_1 \wedge \dots \wedge f_{n-1}) \times P(f_{n-1} | f_1 \wedge \dots \wedge f_{n-2})$$

$$\times \dots \times P(f_3 | f_1 \wedge f_2) \times P(f_2 | f_1) \times P(f_1)$$

$$= \prod_{i=1}^n P(f_i | f_1 \wedge \dots \wedge f_{i-1})$$

E.g., P(weather, temperature) = $P(weather | temperature) \cdot P(temperature).$

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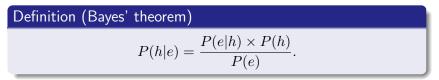
Bayes' theorem

The chain rule and commutativity of conjunction $(h \land e \text{ is equivalent to } e \land h)$ gives us:

$$P(h \wedge e) = P(h|e) \times P(e)$$

= $P(e|h) \times P(h).$

If $P(e) \neq 0$, you can divide the right hand sides by P(e), giving us Bayes' theorem.



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Why is Bayes' theorem interesting?

Often you have causal knowledge:

- $P(symptom \mid disease)$
- *P*(*light is off* | *status of switches and switch positions*)
- $P(alarm \mid fire)$
- $P(\text{image looks like } \blacksquare \mid \text{a tree is in front of a car})$

...and you want to do evidential reasoning:

- $P(disease \mid symptom)$
- $P(\text{status of switches} \mid \text{light is off and switch positions})$
- $P(fire \mid alarm)$.
- $P(a \text{ tree is in front of a car} \mid image looks like <math>\mathbf{A})$

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