Logic: Soundness and Completeness of Bottom-Up Proofs

CPSC 322 – Logic 4

Textbook §5.2

Lecture Overview

1. Recap
2. Soundness of Bottom-Up Proofs
3. Completeness of Bottom-Up Proofs

Proofs

- A proof is a mechanically derivable demonstration that a formula logically follows from a knowledge base.
- Given a proof procedure, $KB \vdash g$ means g can be derived from knowledge base KB.
- Recall $KB \models g$ means g is true in all models of KB.

Definition (soundness)

A proof procedure is sound if $KB \vdash g$ implies $KB \models g$.

Definition (completeness)

A proof procedure is complete if $KB \models g$ implies $KB \vdash g$.

Bottom-up Ground Proof Procedure

One rule of derivation, a generalized form of *modus ponens*:

> *If "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" is a clause in the knowledge base, and each b_i has been derived, then h can be derived.*

You are forward chaining on this clause.
(This rule also covers the case when $m = 0$.)

Bottom-up proof procedure

$KB \vdash g$ if $g \subseteq C$ at the end of this procedure:

$C := \{\}$;
repeat
  select clause "$h \leftarrow b_1 \wedge \ldots \wedge b_m$" in KB such that
    $b_i \in C$ for all i, and $h \notin C$;
  $C := C \cup \{h\}$
until no more clauses can be selected.

Example

$$a \leftarrow b \wedge c.$$
$$a \leftarrow e \wedge f.$$
$$b \leftarrow f \wedge k.$$
$$c \leftarrow e.$$
$$d \leftarrow k.$$
$$e.$$
$$f \leftarrow j \wedge e.$$
$$f \leftarrow c.$$
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Soundness of bottom-up proof procedure

If $KB \vdash g$ then $KB \models g$.

- Suppose there is a g such that $KB \vdash g$ and $KB \not\models g$.
- Let h be the first atom added to C that's not true in every model of KB.
- Suppose h isn't *true* in model I of KB.
- There must be a clause in KB of form

$$h \leftarrow b_1 \wedge \ldots \wedge b_m$$

  Each b_i is true in I. h is false in I. So this clause is false in I.
- Therefore I isn't a model of KB. Contradiction: thus no such g exists.

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- First, observe that the C generated at the end of the bottom-up algorithm is a fixed point.
  - further applications of our rule of derivation will not change C.

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Let the minimal model I be the interpretation in which every element of the fixed point C is true and every other atom is false.

Claim: I is a model of KB. Proof:

- Assume that I is not a model of KB. Then there must exist some clause $h \leftarrow b_1 \wedge \ldots \wedge b_m$ in KB (having zero or more b_i's) which is false in I.
- This can only occur when h is false and each b_i is true in I.
- If each b_i belonged to C, we would have added h to C as well.
- Since C is a fixed point, no such I can exist.

Completeness

If $KB \models g$ then $KB \vdash g$.

- Suppose $KB \models g$. Then g is true in all models of KB.
- Thus g is true in the minimal model.
- Thus g is generated by the bottom up algorithm.
- Thus $KB \vdash g$.