Propositional Logic: Bottom-Up Proofs

CPSC 322 Lecture 20

March 2, 2007 Textbook §4.2

Propositional Logic: Bottom-Up Proofs

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Lecture Overview







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Propositional Definite Clauses: Syntax

Definition (atom)

An atom is a symbol starting with a lower case letter

Definition (body)

A body is an atom or is of the form $b_1 \wedge b_2$ where b_1 and b_2 are bodies.

Definition (definite clause)

A definite clause is an atom or is a rule of the form $h \leftarrow b$ where h is an atom and b is a body. (Read this as "h if b.")

Definition (knowledge base)

A knowledge base is a set of definite clauses

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Propositional Definite Clauses: Semantics

Semantics allows you to relate the symbols in the logic to the domain you're trying to model.

Definition (interpretation)

An interpretation I assigns a truth value to each atom.

We can use the interpretation to determine the truth value of clauses and knowledge bases:

Definition (truth values of statements)

- A body $b_1 \wedge b_2$ is true in I if and only if b_1 is true in I and b_2 is true in I.
- A rule h ← b is false in I if and only if b is true in I and h is false in I.
- A knowledge base *KB* is true in *I* if and only if every clause in *KB* is true in *I*.

Models and Logical Consequence

Definition (model)

A model of a set of clauses is an interpretation in which all the clauses are *true*.

Definition (logical consequence)

If KB is a set of clauses and g is a conjunction of atoms, g is a logical consequence of KB, written $KB \models g$, if g is *true* in every model of KB.

- we also say that g logically follows from KB, or that KB entails g.
- In other words, $KB \models g$ if there is no interpretation in which KB is *true* and g is *false*.

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Propositional Logic: Bottom-Up Proofs

Recap	Proofs	Bottom-Up Proofs
Proofs		

- A proof is a mechanically derivable demonstration that a formula logically follows from a knowledge base.
- Given a proof procedure, $KB \vdash g$ means g can be derived from knowledge base KB.
- Recall $KB \models g$ means g is true in all models of KB.

Definition (soundness)

A proof procedure is sound if $KB \vdash g$ implies $KB \models g$.

Definition (completeness)

A proof procedure is complete if $KB \models g$ implies $KB \vdash g$.

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Bottom-up Ground Proof Procedure

One rule of derivation, a generalized form of modus ponens: If " $h \leftarrow b_1 \land \ldots \land b_m$ " is a clause in the knowledge base, and each b_i has been derived, then h can be derived.

You are forward chaining on this clause. (This rule also covers the case when m = 0.)

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Bottom-up proof procedure

$KB \vdash g$ if $g \subseteq C$ at the end of this procedure:

 $C := \{\};$ repeat
select clause " $h \leftarrow b_1 \land \ldots \land b_m$ " in KB such that $b_i \in C$ for all i, and $h \notin C$; $C := C \cup \{h\}$ until no more clauses can be selected.

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Example

$$a \leftarrow b \land c.$$

$$a \leftarrow e \land f.$$

$$b \leftarrow f \land k.$$

$$c \leftarrow e.$$

$$d \leftarrow k.$$

$$e.$$

$$f \leftarrow j \land e.$$

$$f \leftarrow c.$$

$$j \leftarrow c.$$

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Example

$$\begin{aligned} a &\leftarrow b \wedge c. \\ a &\leftarrow e \wedge f. \\ b &\leftarrow f \wedge k. \\ c &\leftarrow e. \\ d &\leftarrow k. \\ e. \\ f &\leftarrow j \wedge e. \\ f &\leftarrow c. \\ j &\leftarrow c. \end{aligned}$$

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Example

$$a \leftarrow b \land c.$$

$$a \leftarrow e \land f.$$

$$b \leftarrow f \land k.$$

$$c \leftarrow e.$$

$$d \leftarrow k.$$

$$e.$$

$$f \leftarrow j \land e.$$

$$f \leftarrow c.$$

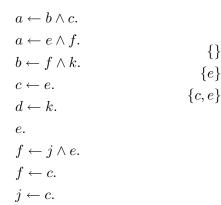
$$j \leftarrow c.$$

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Example

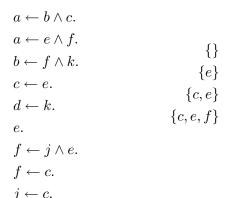


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Example



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Example

$a \leftarrow b \wedge c.$	
$a \leftarrow e \wedge f.$	Ω
$b \leftarrow f \wedge k.$	{} [a]
$c \leftarrow e$.	$\{e\}$ $\{c,e\}$
$d \leftarrow k.$	$\{c, e\}$ $\{c, e, f\}$
е.	
$f \leftarrow j \wedge e.$	$\{c, e, f, j\}$
$f \leftarrow c.$	
$j \leftarrow c.$	

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Example

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$b \leftarrow f \wedge k.$	{} {}
$c \leftarrow e$.	$\{e\}$ $\{c,e\}$
$d \leftarrow k.$	$\{c, e\}$ $\{c, e, f\}$
е.	$\{c, e, f, j\}$ $\{c, e, f, j\}$
$f \leftarrow j \wedge e.$	
$f \leftarrow c.$	$\{a, c, e, f, j\}$
$j \leftarrow c.$	

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Soundness of bottom-up proof procedure

If $KB \vdash g$ then $KB \models g$.

- Suppose there is a g such that $KB \vdash g$ and $KB \not\models g$.
- Let h be the first atom added to C that's not true in every model of KB.
- Suppose h isn't *true* in model I of KB.
- There must be a clause in KB of form

$$h \leftarrow b_1 \land \ldots \land b_m$$

Each b_i is true in I. h is false in I. So this clause is false in I.

• Therefore *I* isn't a model of *KB*. Contradiction: thus no such *g* exists.

Minimal Model

We can use proof procedure to find a model of KB.

- First, observe that the *C* generated at the end of the bottom-up algorithm is a fixed point.
 - further applications of our rule of derivation will not change C.

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Minimal Model

We can use proof procedure to find a model of KB.

- First, observe that the *C* generated at the end of the bottom-up algorithm is a fixed point.
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Let I be the interpretation in which every element of the fixed point C is true and every other atom is false.

• we'll call *I* a minimal model.

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Minimal Model

We can use proof procedure to find a model of KB.

- First, observe that the *C* generated at the end of the bottom-up algorithm is a fixed point.
 - further applications of our rule of derivation will not change C.

Let I be the interpretation in which every element of the fixed point C is true and every other atom is false.

• we'll call *I* a minimal model.

Claim: I is a model of KB. Proof:

- Assume that I is not a model of KB. Then there must exist some clause h ← b₁ ∧ ... ∧ b_m in KB (having zero or more b_i's) which is false in I.
- This can only occur when h is false and each b_i is true in I.
- If each b_i belonged to C, we would have added h to C as well.
- Since C is a fixed point, no such I can exist.

Completeness

If $KB \models g$ then $KB \vdash g$.

- Suppose $KB \models g$. Then g is true in all models of KB.
- Thus g is true in the minimal model.
- Thus g is generated by the bottom up algorithm.
- Thus $KB \vdash g$.

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