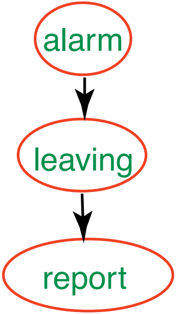
Reasoning Under Uncertainty: Variable Elimination

CPSC 322 Lecture 28

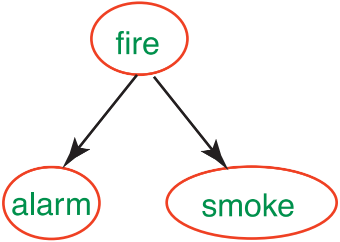
March 20, 2006
Textbook §9.4

Chain



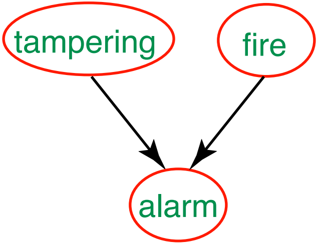
- $alarm$ and $report$ are independent: false.
- $alarm$ and $report$ are independent given $leaving$: true.
- Intuitively, the only way that the $alarm$ affects $report$ is by affecting $leaving$.

Common ancestors



- $alarm$ and $smoke$ are independent: false.
- $alarm$ and $smoke$ are independent given $fire$: true.
- Intuitively, $fire$ can explain $alarm$ and $smoke$; learning one can affect the other by changing your belief in $fire$.

Common descendants



- $tampering$ and $fire$ are independent: true.
- $tampering$ and $fire$ are independent given $alarm$: false.
- Intuitively, $tampering$ can explain away $fire$

Belief Network Inference

- Our goal: compute probabilities of variables in a belief network
- Two cases:
  1. the unconditional (prior) distribution over one or more variables
  2. the posterior distribution over one or more variables, conditioned on one or more observed variables
- To address both cases, we only need a computational solution to case 1
- Our method: exploiting the structure of the network to efficiently eliminate (sum out) the non-observed, non-query variables one at a time.

Evidence

- If we want to compute the posterior probability of Z given evidence $Y_1 = v_1 \land \ldots \land Y_j = v_j$:

$$
\begin{aligned}
& P(Z|Y_1 = v_1, \ldots, Y_j = v_j) \\
= & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{P(Y_1 = v_1, \ldots, Y_j = v_j)} \\
= & \frac{P(Z, Y_1 = v_1, \ldots, Y_j = v_j)}{\sum_Z P(Z, Y_1 = v_1, \ldots, Y_j = v_j).}
\end{aligned}
$$

- So the computation reduces to the probability of $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$.

Factors

- A factor is a representation of a function from a tuple of random variables into a number.
- We will write factor f on variables $X_1, \ldots, X_j$ as $f(X_1, \ldots, X_j)$.
- A factor denotes a distribution over the given tuple of variables in some (unspecified) context
  - e.g., $P(X_1, X_2)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_2, X_3 = v_3)$ is a factor $f(X_1, X_2)$
  - e.g., $P(X_1, X_3 = v_3 | X_2)$ is a factor $f(X_1, X_2)$

Manipulating Factors

- We can make new factors out of an existing factor
- Our first operation: we can assign some or all of the variables of a factor.
  - $f(X_1 = v_1, X_2, \ldots, X_j)$, where $v_1 \in dom(X_1)$, is a factor on $X_2, \ldots, X_j$.
  - $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number that is the value of f when each X_i has value v_i.
- The former is also written as $f(X_1, X_2, \ldots, X_j)_{X_1 = v_1, \ldots, X_j = v_j}$

Example factors

$r(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$r(X{=}t, Y, Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$r(X{=}t, Y, Z{=}f)$:

Y	val
t	0.9
f	0.8

$r(X{=}t, Y{=}f, Z{=}f) = 0.8$

Summing out variables

Our second operation: we can sum out a variable, say X_1 with domain $\{v_1, \ldots, v_k\}$, from factor $f(X_1, \ldots, X_j)$, resulting in a factor on $X_2, \ldots, X_j$ defined by:

$$\begin{aligned}
&(\sum_{X_1} f)(X_2, \ldots, X_j) \\
&\quad = \quad f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)
\end{aligned}$$

Summing out a variable example

f_3:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\sum_B f_3$:

A	C	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

Multiplying factors

- Our third operation: factors can be multiplied together.
- The product of factor $f_1(\overline{X}, \overline{Y})$ and $f_2(\overline{Y}, \overline{Z})$, where $\overline{Y}$ are the variables in common, is the factor $(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z})$ defined by:

$$(f_1 \times f_2)(\overline{X}, \overline{Y}, \overline{Z}) \quad = \quad f_1(\overline{X}, \overline{Y}) f_2(\overline{Y}, \overline{Z}).$$

Multiplying factors example

f_1:

A	B	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

B	C	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$f_1 \times f_2$:

A	B	C	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

Probability of a conjunction

- Suppose the variables of the belief network are $X_1, \ldots, X_n$.
- What we want to compute: the factor $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$
- We can compute $P(Z, Y_1 = v_1, \ldots, Y_j = v_j)$ by summing out the variables $Z_1, \ldots, Z_k = \{X_1, \ldots, X_n\} \setminus \{Z, Y_1, \ldots, Y_j\}$.
- We sum out these variables one at a time
  - the order in which we do this is called our elimination ordering.

$$
\begin{aligned}
& P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
& = \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Probability of a conjunction

- What we know: the factors $P(X_i|pX_i)$.
- Using the chain rule and the definition of a belief network, we can write $P(X_1, \ldots, X_n)$ as $\prod_{i=1}^{n} P(X_i|pX_i)$. Thus:

$$
\begin{aligned}
& P(Z, Y_1 = v_1, \ldots, Y_j = v_j) \\
&= \sum_{Z_k} \cdots \sum_{Z_1} P(X_1, \ldots, X_n)_{Y_1 = v_1, \ldots, Y_j = v_j}. \\
&= \sum_{Z_k} \cdots \sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)_{Y_1 = v_1, \ldots, Y_j = v_j}.
\end{aligned}
$$

Computing sums of products

Computation in belief networks thus reduces to computing the sums of products.

- It takes 14 multiplications or additions to evaluate the expression $ab + ac + ad + aeh + afh + agh$. How can this expression be evaluated more efficiently?
  - factor out the a and then the h giving $a(b + c + d + h(e + f + g))$
  - this takes only 7 multiplications or additions
- How can we compute $\sum_{Z_1} \prod_{i=1}^{n} P(X_i|pX_i)$ efficiently?
- Factor out those terms that don't involve Z_1:

$$\left(\underbrace{\prod_{i|Z_1 \notin \{X_i\} \cup pX_i}}_{\text{(terms that do not involve } Z_i\text{)}} P(X_i|pX_i) \right) \left(\sum_{Z_1} \underbrace{\prod_{i|Z_1 \in \{X_i\} \cup pX_i}}_{\text{(terms that involve } Z_i\text{)}} P(X_i|pX_i) \right)$$

(terms that do not involve Z_i) *(terms that involve Z_i)*

Summing out a variable efficiently

To sum out a variable Z_j from a product $f_1, \ldots, f_k$ of factors:

- Partition the factors into
  - those that don't contain Z_j, say $f_1, \ldots, f_i$,
  - those that contain Z_j, say $f_{i+1}, \ldots, f_k$

We know:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = (f_1 \times \cdots \times f_i) \left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k \right).$$

- $\left(\sum_{Z_j} f_{i+1} \times \cdots \times f_k\right)$ is a new factor; let's call it f'.
- Now we have:

$$\sum_{Z_j} f_1 \times \cdots \times f_k = f_1 \times \cdots \times f_i \times f'.$$

- Store f' explicitly, and discard $f_{i+1}, \ldots, f_k$. Now we've summed out Z_j.

Variable elimination algorithm

To compute $P(Z|Y_1 = v_1 \wedge \ldots \wedge Y_j = v_j)$:

- Construct a factor for each conditional probability.
- Set the observed variables to their observed values.
- For each of the other variables $Z_i \in \{Z_1, \ldots, Z_k\}$, sum out Z_i
- Multiply the remaining factors.
- Normalize by dividing the resulting factor $f(Z)$ by $\sum_Z f(Z)$.