A General Framework for Computing Optimal Correlated Equilibria in Compact Games (Extended Abstract)*

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Abstract. We analyze the problem of computing a correlated equilibrium that optimizes some objective (e.g., social welfare). Papadimitriou and Roughgarden [2008] gave a sufficient condition for the tractability of this problem; however, this condition only applies to a subset of existing representations. We propose a different algorithmic approach for the optimal CE problem that applies to *all* compact representations, and give a sufficient condition that generalizes that of Papadimitriou and Roughgarden [2008]. In particular, we reduce the optimal CE problem to the *deviation-adjusted social welfare problem*, a combinatorial optimization problem closely related to the optimal social welfare problem. This framework allows us to identify new classes of games for which the optimal CE problem is tractable; we show that graphical polymatrix games on tree graphs are one example. We also study the problem of computing the optimal *coarse correlated equilibrium*, a solution concept closely related to CE. Using a similar approach we derive a sufficient condition for this problem, and use it to prove that the problem is tractable for singleton congestion games.

1 Introduction

A fundamental class of computational problems in game theory is the computation of *solution concepts* of finite games. Much recent effort in the literature has concerned the problem of computing a sample Nash equilibrium [Chen & Deng, 2006; Daskalakis *et al.*, 2006; Daskalakis & Papadimitriou, 2005; Goldberg & Papadimitriou, 2006]. First proposed by Aumann [1974; 1987], correlated equilibrium (CE) is another important solution concept. Whereas in a mixed strategy Nash equilibrium players randomize independently, in a correlated equilibrium the players can coordinate their behavior based on signals from an intermediary.

Correlated equilibria of a game can be formulated as probability distributions over pure strategy profiles satisfying certain linear constraints. The resulting linear feasibility program has size polynomial in the size of the normal form representation of the game. However, the size of the normal form representation grows exponentially in the number

* All proofs are omitted in this extended abstract. A full version is available at http://arxiv.org/abs/1109.6064.

of players. This is problematic when games involve large numbers of players. Fortunately, most large games of practical interest have highly-structured payoff functions, and thus it is possible to represent them compactly. A line of research thus exists to look for *compact game representations* that are able to succinctly describe structured games, including work on graphical games [Kearns *et al.*, 2001] and action-graph games [Bhat & Leyton-Brown, 2004; Jiang *et al.*, 2011]. But now the size of the linear feasibility program for CE can be exponential in the size of compact representation; furthermore a CE can require exponential space to specify.

The problem of computing a sample CE was recently shown to be in polynomial time for most existing compact representations [Papadimitriou & Roughgarden, 2008; Jiang & Leyton-Brown, 2011]. However, since in general there can be an infinite number of CE in a game, finding an arbitrary one is of limited value. Instead, here we focus on the problem of computing a correlated equilibrium that optimizes some objective. In particular we consider optimizing linear functions of players' expected utilities. For example, computing the best (or worst) social welfare corresponds to maximizing (or minimizing) the sum of players' utilities, respectively. We are also interested in computing optimal coarse correlated equilibrium (CCE) [Hannan, 1957]. It is known that the empirical distribution of any no-external-regret learning dynamic converges to the set of CCE, while the empirical distribution of no-internal-regret learning dynamics converges to the set of CE (see e.g. [Nisan *et al.*, 2007]). Thus, optimal CE / CCE provide useful bounds on the social welfare of the empirical distributions of these dynamics.

We are particularly interested in the relationship between the optimal CE / CCE problems and the problem of computing the optimal social welfare outcome (i.e. strategy profile) of the game, which is exactly the optimal social welfare CE problem without the incentive constraints. This is an instance of a line of questions that has received much interest from the algorithmic game theory community: "How does adding incentive constraints to an optimization problem affect its complexity?" This question in the mechanism design setting is perhaps one of the central questions of algorithmic mechanism design [Nisan & Ronen, 2001]. Of course, a more constrained problem can in general be computationally easier than the relaxed version of the problem. Nevertheless, results from complexity of Nash equilibria and algorithmic mechanism design suggest that adding *incentive constraints* to a problem is unlikely to decrease its computational difficulty. That is, when the optimal social welfare problem is hard, we tend also to expect that the optimal CE problem will be hard as well. On the other hand, we are interested in the other direction: when it is the case for a class of games that the optimal social welfare problem can be efficiently computed, can the same structure be exploited to efficiently compute the optimal CE?

The seminal work on the computation of optimal CE is [Papadimitriou & Roughgarden, 2008]. This paper considered the optimal linear objective CE problem and proved that the problem is NP-hard for many representations including graphical games, polymatrix games, and congestion games. On the tractability side, Papadimitriou and Roughgarden [2008] focused on so-called "reduced form" representations, meaning representations for which there exist player-specific partitions of the strategy profile space into payoff-equivalent outcomes. They showed that if a particular *separation problem* is polynomial-time solvable, the optimal CE problem is polynomial-time solvable as well. Finally, they showed that this separation problem is polynomial-time solvable for bounded-treewidth graphical games, symmetric games and anonymous games.

Perhaps most surprising and interesting is the *form* of Papadimitriou and Roughgarden's sufficient condition for tractability: their separation problem for an instance of a reduced-form-based representation is essentially equivalent to solving the optimal social welfare problem for an instance of that representation with the same reduced form but possibly different payoffs. In other words, if we have a polynomial-time algorithm for the optimal social welfare problem for a reduced-form-based representation, we can turn that into a polynomial-time algorithm for the optimal social welfare CE problem. However, Papadimitriou and Roughgarden's sufficient condition for tractability only applies to reduced-form-based representations. Their definition of reduced forms is unable to handle representations that exploit linearity of utility, and in which the structure of player p's utility function may depend on the action she chose. As a result, many representations do not fall into this characterization, such as polymatrix games, congestion games, and action-graph games. Although the optimal CE problems for these representations are NP-hard in general, we are interested in identifying tractable subclasses of games, and a sufficient condition that applies to all representations would be helpful.

In this article, we propose a different algorithmic approach for the optimal CE problem that applies to *all* compact representations. By applying the ellipsoid method to the dual of the LP for optimal CE, we show that the polynomial-time solvability of what we call the *deviation-adjusted social welfare problem* is a sufficient condition for the tractability of the optimal CE problem. We also give a sufficient condition for tractability of the optimal CCE problem: the polynomial-time solvability of the *coarse deviation-adjusted social welfare problem*. We show that for reduced-form-based representations, the deviation-adjusted social welfare problem can be reduced to the separation problem of Papadimitriou and Roughgarden [2008]. Thus the class of reduced forms for which our problem is polynomial-time solvable contains the class for which the separation problem is polynomial-time solvable. More generally, we show that if a representation can be characterized by "linear reduced forms", i.e. player-specific linear functions over partitions, then for that representation, the deviation-adjusted social welfare problem can be reduced to the optimal social welfare problem. As an example, we show that for graphical polymatrix games on trees, optimal CE can be computed in polynomial time. Such games are not captured by the reduced-form framework.[1]

On the other hand, representations like action-graph games and congestion games have *action-specific* structure, and as a result the deviation-adjusted social welfare problems and coarse deviation-adjusted social welfare problems on these representations are structured differently from the corresponding optimal social welfare problems. Nevertheless, we are able to show a polynomial-time algorithm for the optimal CCE problem on *singleton congestion games* [Ieong *et al.*, 2005], a subclass of congestion games. We use a symmetrization argument to reduce the optimal CCE problem to the coarse deviation-adjusted social welfare problem with player-symmetric deviations, which can

[1] In a recent paper Kamisetty *et al.* [2011] has independently proposed an algorithm for optimal CE in graphical polymatrix games on trees. They used a different approach that is specific to graphical games and graphical polymatrix games, and it is not obvious whether their approach can be extended to other classes of games.

be solved using a dynamic-programming algorithm. This is an example where the optimal CCE problem is tractable while the complexity of the optimal CE problem is not yet known.

2 Problem Formulation

Consider a simultaneous-move game $G = (\mathcal{N}, \{S_p\}_{p\in\mathcal{N}}, \{u^p\}_{p\in\mathcal{N}})$, where $\mathcal{N} = \{1,\ldots,n\}$ is the set of players. Denote a player p, and player p's set of pure strategies (i.e., actions) S_p. Let $m = \max_p |S_p|$. Denote a pure strategy profile $s = (s_1,\ldots,s_n) \in S$, with s_p being player p's pure strategy. Denote by S_{-p} the set of partial pure strategy profiles of the players other than p. Let u^p be the vector of player p's utilities for each pure profile, denoting player p's utility under pure strategy profile s as u^p_s. Let w be the vector of social welfare for each pure profile, that is $w = \sum_{p\in\mathcal{N}} u^p$, with w_s denoting the social welfare for pure profile s.

Throughout the paper we assume that the game is given in a representation with *polynomial type* [Papadimitriou, 2005; Papadimitriou & Roughgarden, 2008], i.e., that the number of players and the number of actions for each player are bounded by polynomials of the size of the representation.

2.1 Correlated Equilibrium

A *correlated distribution* is a probability distribution over pure strategy profiles, represented by a vector $x \in \mathbb{R}^M$, where $M = \prod_p |S_p|$. Then x_s is the probability of pure strategy profile s under the distribution x.

Definition 1. *A correlated distribution x is a* correlated equilibrium *(CE) if it satisfies the following* incentive constraints*: for each player p and each pair of her actions $i, j \in S_p$, we have $\sum_{s_{-p}\in S_{-p}} [u^p_{is_{-p}} - u^p_{js_{-p}}] x_{is_{-p}} \geq 0$, where the subscript "$is_{-p}$" (respectively "$js_{-p}$") denotes the pure strategy profile in which player p plays i (respectively j) and the other players play according to the partial profile $s_{-p} \in S_{-p}$.*

Intuitively, when a trusted intermediary draws a strategy profile s from this distribution, privately announcing to each player p her own component s_p, p will have no incentive to choose another strategy, assuming others follow the suggestions. We write these incentive constraints in matrix form as $Ux \geq 0$. Thus U is an $N \times M$ matrix, where $N = \sum_p |S_p|^2$. The rows of U are indexed by (p, i, j), where p is a player and $i, j \in S_p$ are a pair of p's actions. Denote by U_s the column of U corresponding to pure strategy profile s. These incentive constraints, together with the constraints $x \geq 0,\ \sum_{s\in S} x_s = 1$, which ensure that x is a probability distribution, form a linear feasibility program that defines the set of CE. The problem of computing a maximum social welfare CE can be formulated as the LP

$$\max w^T x \tag{P}$$
$$Ux \geq 0,\ x \geq 0,\ \sum_{s\in S} x_s = 1$$

Another solution concept of interest is *coarse correlated equilibrium* (CCE). Whereas CE requires that each player has no profitable deviation even if she takes into account the signal she receives from the intermediary, CCE only requires that each player has no profitable *unconditional deviation*.

Definition 2. *A correlated distribution x is a* coarse correlated equilibrium *(CCE) if it satisfies the following incentive constraints: for each player p and each of his actions $j \in S_p$, we have $\sum_{(i,s_{-p}) \in S} [u^p_{is_{-p}} - u^p_{js_{-p}}] x_{is_{-p}} \geq 0$.*

We write these incentive constraints in matrix form as $Cx \geq 0$. Thus C is an $(\sum_p |S_p|) \times M$ matrix. By definition, a CE is also a CCE.

The problem of computing a maximum social welfare CCE can be formulated as the LP

$$\max w^T x \tag{CP}$$
$$Cx \geq 0,\ x \geq 0,\ \sum_{s \in S} x_s = 1.$$

3 The Deviation-Adjusted Social Welfare Problem

Consider the dual of (P),

$$\begin{aligned} &\min t \\ U^T y + w &\leq t\mathbf{1} \\ y &\geq 0. \end{aligned} \tag{D}$$

We label the (p, i, j)-th element of $y \in \mathbb{R}^N$ (corresponding to row (p, i, j) of U) as $y^p_{i,j}$. This is an LP with a polynomial number of variables and an exponential number of constraints. Given a separation oracle, we can solve it in polynomial time using the ellipsoid method. A separation oracle needs to determine whether a given (y, t) is feasible, and if not output a hyperplane that separates (y, t) from the feasible set. We focus on a restricted form of separation oracles, which outputs a violated constraint for infeasible points.[2] Such a separation oracle needs to solve the following problem:

Problem 1. Given (y, t) with $y \geq 0$, determine if there exists an s such that $(U_s)^T y + w_s > t$; if so output such an s.

The left-hand-side expression $(U_s)^T y + w_s$ is the social welfare at s plus the term $(U_s)^T y$. Observe that the (p, i, j)-th entry of U_s is $u^p_s - u^p_{js_{-p}}$ if $s_p = i$ and is zero otherwise. Thus $(U_s)^T y = \sum_p \sum_{j \in S_p} y^p_{s_p,j} \left(u^p_s - u^p_{js_{-p}} \right)$. We now reexpress $(U_s)^T y + w_s$ in terms of *deviation-adjusted utilities* and *deviation-adjusted social welfare*.

[2] This is a restriction because in general there exist separating hyperplanes other than the violated constraints. For example Papadimitriou and Roughgarden [2008]'s algorithm for computing a sample CE uses a separation oracle that outputs a convex combination of the constraints as a separating hyperplane.

Definition 3. *Given a game, and a vector $y \in \mathbb{R}^N$ such that $y \geq 0$, the* deviation-adjusted utility *for player p under pure profile s is*

$$\hat{u}^p_s(y) = u^p_s + \sum_{j \in S_p} y^p_{s_p,j} \left(u^p_s - u^p_{js_{-p}} \right).$$

The deviation-adjusted social welfare is $\hat{w}_s(y) = \sum_p \hat{u}^p_s(y)$.

By construction, the deviation-adjusted social welfare $\hat{w}_s(y) = \sum_p u^p_s + \sum_p \sum_{j \in S_p} y^p_{s_p,j} \left(u^p_s - u^p_{js_{-p}} \right) = (U_s)^T y + w_s$. Therefore, Problem 1 is equivalent to the following *deviation-adjusted social welfare problem*.

Definition 4. *For a game representation, the* deviation-adjusted social welfare problem *is the following: given an instance of the representation and rational vector $(y,t) \in \mathbb{Q}^{N+1}$ such that $y \geq 0$, determine if there exists an s such that the deviation-adjusted social welfare $\hat{w}_s(y) > t$; if so output such an s.*

Proposition 1. *If the deviation-adjusted social welfare problem can be solved in polynomial time for a game representation, then so can the problem of computing the maximum social welfare CE.*

Let us consider interpretations of the dual variables y and the deviation-adjusted social welfare of a game. The dual (D) can be rewritten as $\min_{y \geq 0} \max_s \tilde{w}_s(y)$. By weak duality, for a given $y \geq 0$ the maximum deviation-adjusted social welfare $\max_s \tilde{w}_s(y)$ is an upper bound on the maximum social welfare CE. So the task of the dual (D) is to find y such that the resulting maximum deviation-adjusted social welfare gives the tightest bound.[3] At optimum, y corresponds to the concept of "shadow prices" from optimization theory; that is, y^p_{ij} equals the rate of change in the social welfare objective when the constraint (p,i,j) is relaxed infinitesimally. Compared to the maximum social welfare CE problem, the maximum deviation-adjusted social welfare problem replaces the incentive constraints with a set of additional penalties or rewards. Specifically, we can interpret y as a set of nonnegative prices, one for each incentive constraint (p,i,j) of (P). At strategy profile s, for each incentive constraint (p,i,j) we impose a penalty equal to y^p_{ij} times the amount the constraint (p,i,j) is violated by s. Note that the penalty can be negative, and is zero if $s_p \neq i$. Then $\tilde{w}_s(y)$ is equal to the social welfare of the modified game.

Practical computation. The problem of computing the expected utility (EU) given a mixed strategy profile has been established as an important subproblem for both the sample NASH problem and the sample CE problem, both in theory [Daskalakis *et al.*, 2006; Papadimitriou & Roughgarden, 2008] and in practice [Blum *et al.*, 2006; Jiang *et al.*, 2011]. Our results suggest that the deviation-adjusted social welfare problem is of similar importance to the optimal CE problem. This connection is more than theoretical: our algorithmic approach can be turned into a practical method for computing optimal CE. In particular, although it makes use of the ellipsoid method, we can easily

[3] An equivalent perspective is to view y as Lagrange multipliers, and the optimal deviation-adjusted SW problem as the Lagrangian relaxation of (P) given the multipliers y.

substitute a more practical method, such as simplex with column generation. In contrast, Papadimitriou and Roughgarden [2008]'s algorithmic approach for reduced forms makes two nested applications of the ellipsoid method, and is less likely to be practical.

3.1 The Coarse Deviation-Adjusted Social Welfare Problem

For the optimal social welfare CCE problem, we can form the dual of (CP)

$$\begin{aligned} &\min t \\ C^T y + w &\leq t\mathbf{1} \\ y &\geq 0 \end{aligned} \tag{1}$$

Definition 5. *We label the (p, j)-th element of y as y_j^p. Given a game, and a vector $y \in \mathbb{R}^{\sum_p |S_p|}$ such that $y \geq 0$, the* coarse deviation-adjusted utility *for player p under pure profile s is $\tilde{u}_s^p(y) = u_s^p + \sum_{j \in S_p} y_j^p (u_s^p - u_{js_{-p}}^p)$. The coarse deviation-adjusted social welfare is $\tilde{w}_s(y) = \sum_p \tilde{u}_s^p(y)$.*

Proposition 2. *If the coarse deviation-adjusted social welfare problem can be solved in polynomial time for a game representation, then the problem of computing the maximum social welfare CCE is in polynomial time for this representation.*

The coarse deviation-adjusted social welfare problem reduces to the deviation-adjusted social welfare problem. To see this, given an input vector y for the coarse deviation-adjusted social welfare problem, we can construct an input vector $y' \in \mathbb{Q}^N$ for the deviation-adjusted social welfare problem with $y'^p_{ij} = y^p_j$ for all $p \in \mathcal{N}$ and $i, j \in S_p$.

4 The Deviation-Adjusted Social Welfare Problem for Specific Representations

In this section we study the deviation-adjusted social welfare problem and its variants on specific representations. Depending on the representation, the deviation-adjusted social welfare problem is not always solvable in polynomial time. Indeed, Papadimitriou and Roughgarden [2008] showed that for many representations the problem of optimal CE is NP-hard. Nevertheless, for such representations we can often identify tractable subclasses of games. We will argue that the deviation-adjusted social welfare problem is a more useful formulation for identifying tractable classes of games than the separation problem formulation of Papadimitriou and Roughgarden [2008], as the latter only applies to reduced-form-based representations.

4.1 Reduced Forms

Papadimitriou and Roughgarden [2008] gave the following reduced form characterization of representations.

Definition 6 **([Papadimitriou & Roughgarden, 2008]).** *Consider a game $G = (\mathcal{N}, \{S_p\}_{p\in\mathcal{N}}, \{u^p\}_{p\in\mathcal{N}})$. For $p = 1, \ldots, n$, let $P_p = \{C_p^1 \ldots C_p^{r_p}\}$ be a partition of S_{-p} into r_p classes. The set $\mathcal{P} = \{P_1, \ldots, P_n\}$ of partitions is a* reduced form *of G if $u_s^p = u_{s'}^p$ whenever (1) $s_p = s'_p$ and (2) both s_{-p} and s'_{-p} belong to the same class in P_p. The* size *of a reduced form is the number of classes in the partitions plus the bits required to specify a payoff value for each tuple (p, k, ℓ) where $1 \le p \le n$, $1 \le k \le r_p$ and $\ell \in S_p$.*

Intuitively, the reduced form imposes the condition that p's utility for choosing an action s_p depends only on which *class* in the partition P_p the profile of the others' actions belongs to. Papadimitriou and Roughgarden [2008] showed that several compact representations such as graphical games and anonymous games have natural reduced forms whose sizes are (roughly) equal to the sizes of the representation. We say such a compact representation has a *concise reduced form*. Intuitively, such a reduced form describes the structure of the game's utility functions.

Let $\mathcal{S}_p(k, \ell)$ denote the set of pure strategy profiles s such that $s_p = \ell$ and s_{-p} is in the k-th class C_p^k of P_p, and let $u^p_{(k,\ell)}$ denote the utility of p for that set of strategy profiles. Papadimitriou and Roughgarden [2008] defined the following *Separation Problem* for a reduced form.

Definition 7 **([Papadimitriou & Roughgarden, 2008]).** *Let $\mathcal{P}$ be a reduced form for game G. The* Separation Problem *for $\mathcal{P}$ is the following: Given rational numbers $\gamma_p(k, \ell)$ for all $p \in \{1, \ldots, n\}$, $k \in \{1, \ldots, r_p\}$, and $\ell \in S_p$, is there a pure strategy profile s such that $\sum_{p,k,\ell:s\in\mathcal{S}_p(k,\ell)} \gamma_p(k, \ell) < 0$? If so, find such s.*

Since $s \in \mathcal{S}_p(k, \ell)$ implies $s_p = \ell$, the left-hand side of the above expression is equivalent to $\sum_p \sum_{k:s\in\mathcal{S}_p(k,s_p)} \gamma_p(k, s_p)$. Furthermore, since s belongs to exactly one class in P_p, the expression is a sum of exactly n summands.

Papadimitriou and Roughgarden [2008] proved that if the separation problem can be solved in polynomial time, then a CE that maximizes a given linear objective in the players' utilities can be computed in time polynomial in the size of the reduced form. How does Papadimitriou and Roughgarden [2008]'s sufficient condition relate to ours, provided that the game has a concise reduced form? We show that the class of reduced form games for which our deviation-adjusted social welfare problem is polynomial-time solvable contains the class for which the separation problem is polynomial-time solvable.

Proposition 3. *Let $\mathcal{P}$ be a reduced form for game G. Suppose the separation problem can be solved in polynomial time. Then the deviation-adjusted social welfare problem can be solved in time polynomial in the size of the reduced form.*

We now compare the the deviation-adjusted social welfare problem with the optimal social welfare problem for these representations. We observe that the deviation-adjusted social welfare problem can be formulated as an instance of the optimal social welfare problem on another game with the same reduced form but different payoffs. Can we claim that the existence of a polynomial-time algorithm for the optimal social welfare problem for a representation implies the existence of a polynomial-time algorithm for the social welfare problem (and thus the optimal CE problem)? This is not necessarily the case, because the representation might impose certain structure on the utility functions that are not captured by the reduced forms, and the polynomial-time algorithm for the optimal social welfare problem could depend on the existence of such structure. The deviation-adjusted social welfare problem might no longer exhibit such structure and thus might not be solvable using the given algorithm.

Nevertheless, if we consider a game representation that is "completely characterized" by its reduced forms, the deviation-adjusted social welfare problem is equivalent to the decision version of the optimal social welfare outcome problem for that representation. To make this more precise, we say a game representation is a *reduced-form-based representation* if there exists a mapping from instances of the representation to reduced forms such that it maps each instance to a concise reduced form of that instance, and if we take such a reduced form and change its payoff values arbitrarily, the resulting reduced form is a concise reduced form of another instance of the representation.

Corollary 1. *For a reduced-form-based representation, if there exists a polynomial-time algorithm for the optimal social welfare problem, then the optimal social welfare CE problem and the max-min welfare CE problem can be solved in polynomial time.*

Of course, this can be derived using the separation problem for reduced forms without the deviation-adjusted social welfare formulation. On the other hand, the deviation-adjusted social welfare formulation can be applied to representations without concise reduced forms. In fact, we will use it to show below that the connection between the optimal social welfare problem and the optimal CE problem applies to a wider classes of representations than just reduced-form-based representations.

4.2 Linear Reduced Forms

One class of representations that does not have concise reduced forms are those that represent utility functions as sums of other functions, such as polymatrix games and the hypergraph games of Papadimitriou and Roughgarden [2008]. In this section we characterize these representations using linear reduced forms, showing that linear-reduced-form-based representations satisfy a property similar to Corollary 1.

Roughly speaking, a linear reduced form has multiple partitions for each agent, rather than just one; an agent's overall utility is a sum over utility functions defined on each of that agent's partitions.

Definition 8. *Consider a game* $G = (\mathcal{N}, \{S_p\}_{p\in\mathcal{N}}, \{u^p\}_{p\in\mathcal{N}})$*. For* $p = 1, \ldots, n$*, let* $P_p = \{P_{p,1}, \ldots, P_{p,t_p}\}$*, where* $P_{p,q} = \{C^1_{p,q} \ldots C^{r_{pq}}_{p,q}\}$ *is a partition of* S_{-p} *into* r_{pq} *classes. The set* $\mathcal{P} = \{P_1, \ldots, P_n\}$ *is a* linear reduced form *of* G *if for each* p *there exist* $u^{p,1}, \ldots, u^{p,t_p} \in \mathbb{R}^M$ *such that for all* s*,* $u^p_s = \sum_q u^{p,q}_s$*, and for each* $q \le t_p$*,* $u^{p,q}_s = u^{p,q}_{s'}$ *whenever (1)* $s_p = s'_p$ *and (2) both* s_{-p} *and* s'_{-p} *belong to the same class in* $P_{p,q}$*. The* size *of a reduced form is the number of classes in the partitions plus the bits required to specify a number for each tuple* (p, q, k, ℓ) *where* $1 \le p \le n$*,* $1 \le q \le t_p$*,* $1 \le k \le r_{pq}$ *and* $\ell \in S_p$*.*

We write $u^{p,q}_{(k,\ell)}$ for the value corresponding to tuple (p, q, k, ℓ), and for $\mathbf{k} = (k_1, \ldots, k_{t_p})$ we write $u^p_{(\mathbf{k},\ell)} \equiv \sum_q u^{p,q}_{(k_q,\ell)}$.

Example 1 (polymatrix games). In a polymatrix game, each player's utility is the sum of utilities resulting from her bilateral interactions with each of the $n-1$ other players: $u_s^p = \sum_{p' \neq p} e_{s_p}^T A^{pp'} e_{s_{p'}}$ where $A^{pp'} \in \mathbb{R}^{|S_p| \times |S_{p'}|}$ and $e_{s_p} \in \mathbb{R}^{|S_p|}$ is the unit vector corresponding to s_p. The utility functions of such a representation require only $\sum_{p,p' \in \mathcal{N}} |S_p| \times |S_{p'}|$ values to specify. Polymatrix games do not have a concise reduced-form encoding, but can easily be written as linear-reduced-form games. Essentially, we create one partition for every matrix game that an agent plays, with each class differing in the action played by the other agent who participates in that matrix game, and containing all the strategy profiles that can be adopted by all of the other players. Formally, given a polymatrix game, we construct its linear reduced form with $P_p = \{P_{p,q}\}_{q \in \mathcal{N} \setminus \{p\}}$, and $P_{p,q} = \{C_{p,q}^{\ell}\}_{\ell \in S_q}$ with $C_{p,q}^{\ell} = \{s_{-p} | s_q = \ell\}$. □

Most of the results in Section 4.1 straightforwardly translate to linear reduced forms.

Corollary 2. *For a linear-reduced-form-based representation, if there exists a polynomial-time algorithm for the optimal social welfare problem, then the optimal social welfare CE problem and the max-min welfare CE problem can be solved in polynomial time.*

Graphical Polymatrix Games A polymatrix game may have graphical-game-like structure: player p's utility may depend only on a subset of the other player's actions. In terms of utility functions, this corresponds to $A^{pp'} = 0$ for certain pairs of players p, p'. As with graphical games, we can construct the (undirected) graph $G = (\mathcal{N}, E)$ where there is an edge $\{p, p'\} \in E$ if $A^{pp'} \neq 0$ or$A^{p'p} \neq 0$. We call such a game a graphical polymatrix game. This can also be understood as a graphical game where each player p's utility is the sum of bilateral interactions with her neighbors.

A tree polymatrix game is a graphical polymatrix game whose corresponding graph is a tree. Consider the optimal CE problem on tree polymatrix games. Since such a game is also a tree graphical game, Papadimitriou and Roughgarden [2008]'s optimal CE algorithm for tree graphical games can be applied. However, this algorithm does not run in polynomial time, because the representation size of tree polymatrix games can be exponentially smaller than that of the corresponding graphical game (which grows exponentially in the degree of the graph). Nevertheless, we give an polynomial-time algorithm for the deviation-adjusted social welfare problem for such games, which then implies the following theorem.

Theorem 1. *Optimal CE in tree polymatrix games can be computed in polynomial time.*

4.3 Representations with Action-Specific Structure

The above results for reduced forms and linear reduced forms crucially depend on the fact that the partitions (i.e., the structure of the utility functions) depend on p but do not depend on the action chosen by player p. There are representations whose utility functions have action-dependent structure, including congestion games [Rosenthal, 1973], local effect games [Leyton-Brown & Tennenholtz, 2003], and action-graph games [Jiang *et al.*, 2011]. For such representations, we can define a variant of the reduced form that has action-dependent partitions. However, unlike both the reduced form and linear reduced form, the deviation-adjusted utilities no longer satisfy the same partition structure as the utilities. Intuitively, the deviation-adjusted utility at s has contributions from the utilities of the strategy profiles when player p deviates to different actions. Whereas for linear reduced forms these deviated strategy profiles correspond to the same class as s in the partition, we now consider different partitions for each action to which p deviates. As a result the deviation-adjusted social welfare problem has a more complex form that the optimal social welfare problem.

Singleton Congestion Games Ieong *et al.* [2005] studies a class of games called singleton congestion games and showed that the optimal PSNE can be computed in polynomial time. Such a game can be formulated as an instance of congestion games where each action contains a single resource, or an instance of symmetric AGGs where the only edges are self edges.

Formally, a singleton congestion game is specified by $(\mathcal{N}, \mathcal{A}, \{f^\alpha\}_{\alpha \in \mathcal{A}})$ where $\mathcal{N} = 1, \ldots, n$ is the set of players, $\mathcal{A}$ the set of actions, and for each action $\alpha \in \mathcal{A}$, $f^\alpha : [n] \to \mathbb{R}$. The game is symmetric; each player's set of actions $S_p \equiv \mathcal{A}$. Each strategy profile s induces an action count $c(\alpha) = |\{p|s_p = \alpha\}|$ on each α: the number of players playing action α. Then the utility of a player that chose α is $f^\alpha(c(\alpha))$. The representation requires $O(|\mathcal{A}|n)$ numbers to specify.

Before attacking the optimal social welfare CCE problem, we first note that the optimal social welfare problem can be solved in polynomial time by a relatively straightforward dynamic-programming algorithm which is a simplified version of Ieong *et al.* [2005]'s algorithm for optimal PSNE in singleton congestion games. Can we leverage the algorithm for the optimal social welfare problem to solve the coarse deviation-adjusted social welfare problem? Our task here is slightly more complicated: in general the coarse deviation-adjusted social welfare problem no longer has the same symmetric structure due to the fact that y can be asymmetric. However, when y is player-symmetric (that is, $y_j^p = y_j^{p'}$ for all pairs of players (p, p')), then we recover symmetric structure.

Lemma 1. *Given a singleton congestion game and player-symmetric input y, the coarse deviation-adjusted social welfare problem can be solved in polynomial time.*

Therefore if we can guarantee that during a run of ellipsoid method for (1) all input queries y to the separation oracle are symmetric, then we can apply Lemma 1 to solve the problem in polynomial time. We observe that for any symmetric game, there must exist a *symmetric* CE that optimizes the social welfare. This is because given an optimal CE we can create a mixture of permuted versions of this CE, which must itself be a CE by convexity, and must also achieve the same social welfare by symmetry. However, this argument in itself does not guarantee that the y we obtain by the method above will be symmetric. Instead, we observe that if we solve (1) using a ellipsoid method with a player-symmetric initial ball, and use a separation oracle that returns a player-symmetric cutting plane, then the query points y will be player-symmetric. We are able to construct such a separation oracle using a symmetrization argument.

Theorem 2. *Given a singleton congestion game, the optimal social welfare CCE can be computed in polynomial time.*

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