Dijkstra-like Ordered Upwind Methods for Solving Static Hamilton-Jacobi Equations

by

Ken Alton

Computer Science

University of British Columbia
Presentation Outline

• Introduction
  – Static Hamilton-Jacobi PDE
  – Approximate Solution
  – Contributions
• Related Methods
• FMM for Axis-Aligned Problems
• Monotone Acceptance OUM
• Summary
Example: Minimum Time to Reach

- Robot to reach goal in minimal time while avoiding obstacles
Problem Formulation

• Dirichlet problem for a static Hamilton-Jacobi PDE:
  \[ H(x, Du(x)) = 0, \quad x \in \Omega \]
  \[ u(x) = g(x), \quad x \in \partial \Omega \]

• Viscosity solution

• Control-theoretic Hamiltonian:
  \[ H(x, q) = \max_{a \in \mathcal{A}} [(-q \cdot a) f(x, a)] - 1 \]

• Unit vector controls:
  \[ \mathcal{A} = \{ a \in \mathbb{R}^d \mid \|a\| = 1 \} \]

• Speed profile:
  \[ \mathcal{A}_f(x) = \{ af(x, a) \mid a \in \mathbb{R}^d \text{ and } \|a\| \leq 1 \} \]
Anisotropy

- Isotropic problem:
  \[ H(x, q) = \max_{a \in \mathcal{A}} [(-q \cdot a)f(x)] - 1 = \|q\|_2 f(x) - 1 \]

  - Eikonal Equation:
    \[ \|Du(x)\|_2 = \frac{1}{f(x)}, \quad x \in \Omega \]
    \[ u(x) = g(x), \quad x \in \partial \Omega \]

- Anisotropic speed profiles:

- Local anisotropy coefficient: \( \gamma(x) = \hat{f}(x)/\tilde{f}(x) \)
Optimal Control

- Follow an optimal trajectory by choosing action that ensures the \textit{fastest descent} of viscosity solution
- Continuous dynamic programming
- A greedy choice is an optimal choice

\[ \hat{a}(x) \in \arg\max_{a \in A}[(-Du(x) \cdot a)f(x,a)] \]
Approximate Solution

• Discretize domain and boundary
  – Structured, semi-structured, and unstructured grids
• Number of nodes $N$, grid spacing $h$
• Approximate the viscosity solution on the resulting grid
• Large system of nonlinear equations
  – How can we solve efficiently?

[Maubach, JSC, 1995]
Dijkstra-like Dynamic Programming

• Causality: a node’s solution value is dependent only on smaller values in stencil

• Solution values are computed in nondecreasing order in a single pass through the grid nodes

```plaintext
foreach \( x \in \Omega \) do \( v(x) \leftarrow \infty \)
foreach \( x \in \partial \Omega \) do \( v(x) \leftarrow g(x) \)
\( \mathcal{H} \leftarrow \mathcal{X} \)
while \( \mathcal{H} \neq \emptyset \) do
  \( x \leftarrow \arg\min_{y \in \mathcal{H}} v(y) \)
  \( \mathcal{H} \leftarrow \mathcal{H} \setminus \{x\} \)
  foreach \( y \in \gamma(x) \cap \mathcal{H} \) do
    \( v(y) \leftarrow \text{Update}(y) \)
  end
end

• Binary heap yields \( O(N \log N) \) complexity
Contributions

• Extend Dijkstra-like methods to anisotropic static HJ PDEs
• Fast Marching Method (FMM) for axis-aligned HJ PDEs
  – Define class of axis-aligned HJ PDEs and compare to Osher’s criterion
  – Prove FMM is applicable to solving this class
  – Discuss efficient implementation
• Dijkstra-like Ordered Upwind Method (OUM) for general convex HJ PDEs
  – Define and prove criteria to ensure extended stencil results in causal discretization
  – Implement method, including stencil construction in initial pass
  – Compare method to original OUM
• Apply methods to robotic minimum-time path planning and seismic wave-front first arrival time problems
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• Introduction
• Related Methods
  – Fast Marching Method (FMM)
  – Ordered Upwind Method (OUM)
  – Sweeping Methods
• FMM for Axis-Aligned Problems
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Fast Marching Method

• Label-setting/single-pass method
• Different Dijkstra-like algorithm depending on Update()
  – Dijkstra’s Algorithm on discrete graph [Dijkstra, Num. Math., 1959]
  \[ v(x) = \min_{y \in \mathcal{N}(x)} \left\{ v(y) + \frac{||y - x||}{f(x)} \right\} \]

  – Dijkstra-like method for Eikonal equation with semi-Lagrangian discretization [Tsitsiklis, CDC, 1994]
  \[ v(x) = \min_{s \in S(x)} \min_{\zeta \in [0,1]} \left\{ \frac{||\tilde{x}_s(\zeta) - x||}{f(x)} + \zeta v(x_1^s) + (1 - \zeta) v(x_2^s) \right\} \]

  – Fast Marching Method with upwind Eulerian finite-difference discretization [Sethian, PNAS, 1996]
Ordered Upwind Method (OUM)

- Modification of FMM to solve problems with general convex speed profiles in $O(N \log N)$
- Update() looks beyond immediate neighbors to use virtual simplices that include nodes within $h \mathcal{R}$
- Maintain a front of accepted nodes
- Accepted Front OUM (AFOUM)
Sweeping Methods

• Label-correcting
• Iterative Gauss-Seidel (GS) algorithms
• GS converges quickly if node order aligned with characteristics
• Alternate among a set of static node orderings
• $O(N)$ but constant problem dependent
• Examples:
  – Markov chain approximations: [Boue & Dupuis, SINUM, 1999]
  – A class of static HJ PDEs: [Tsai et al., SINUM, 2003]
  – Static convex HJ PDEs: [Qian, Zhang, & Zhao, JSC, 2007]
Marching/Sweeping Hybrid Methods

- Approximate the optimal node ordering of Dijkstra-like methods using an untidy priority queue
- Label-setting
  - $O(N)$ method using bucket sort [Yatziv et al., JCP, 2006]
  - Authors claim sort error on order of discretization error
- Label-correcting methods with dynamic node orderings
  - [Polymenakos et al., Trans. Automat. Control, 1998]
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• Introduction
• Related Methods
• FMM for Axis-Aligned Problems
  – Class of Hamiltonians
  – Discretization
  – Efficient Implementation
  – Examples
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FMM for Axis-aligned Problems

- FMM works for axis-aligned anisotropic problems on orthogonal grids
- Uses Eulerian upwind finite-difference discretization
Class of Hamiltonians

- **Gradient:** \( q = Du(x) \)
- **Assume** \( H \) satisfies
  - Continuity: \( H \in C(\Omega \times \mathbb{R}^d) \)
  - Coercivity: For all \( x \in \Omega \), \( H(x, q) \to \infty \) as \( \|q\| \to \infty \)
  - Strict compatibility: For all \( x \in \Omega \), \( H(x, 0) < 0 \)
  - Strict one-sided monotonicity:
    For all \( x \in \Omega \), if \( q > \tilde{q} \), then \( H(x, q) > H(x, \tilde{q}) \)
- **Osher’s criterion** [Osher & Fedkiw, JCP, 2001]:
  \[
  q_j \frac{\partial H(x, q)}{\partial q_j} \geq 0
  \]
Examples

- Hamiltonian: $H(x, q) = G(x, q) - 1$

\[ p = 2 \quad p = \infty \quad p = 1 \quad p = 1 \text{ (scaled)} \]

asymmetric
\[ \|((q_1, q_2)\|_2, q_3)\|_1 \]
not one-sided monotone
Eulerian Discretization

- Numerical Hamiltonian:
  \[ H(x, N, \phi, \mu) = \max_{\kappa \in \mathcal{K}} [H(x, D^\kappa(x, N, \phi, \mu))] \]

- Upwind Eulerian finite-difference gradient approximation:
  \[
  D_j^\kappa(x, N, \phi, \mu) = \begin{cases} 
  \frac{\max(0, \mu - \phi(x_j^{\kappa j}))}{-h_j^{\kappa j}}, & \text{if } x_j^{\kappa j} \in N \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Update equation:
  \[ H(x, N(x) \setminus \mathcal{H}, v, v(x)) = 0 \]
Convergence and Applicability of FMM

\[
H(x, N, \phi, \mu) = \max_{\kappa \in \mathcal{K}} [H(x, D^\kappa(x, N, \phi, \mu))]
\]

\[
D^\kappa_{ij}(x, N, \phi, \mu) = \begin{cases} 
\frac{\max(0, \mu - \phi(x^{\kappa j}_j))}{-h^{\kappa j}_j}, & \text{if } x^{\kappa j}_j \in N \\
0, & \text{otherwise}
\end{cases}
\]

  - Monotonicity: if none of stencil node’s values decrease then the numerical Hamiltonian should not increase
  - Results from finite-difference approximation of gradient and one-sided monotonicity of Hamiltonian

- Causality necessary for applicability of FMM
  - A node’s solution value is dependent only on smaller values
  - Results from the upwind approximation of the gradient
Update Efficiency

• Update() will be called several times for each node
  – For each call to Update(), a solution to $H(\mu) = 0$ must be computed for each orthant
  – Analytic solution to $H(\mu) = 0$ for an orthant is available for standard $p$-norms with any grid spacing
  – General problems require numerical solution

• How to eliminate orthants from consideration without solving?
  – Symmetry node elimination
  – Causality node elimination
  – Orthant solution elimination
Example: Asymmetric Anisotropy

- Distance to the origin subject to walls
- Homogenous but anisotropic motion defined by $H(x, q) = G(x, q) - 1$

$$G(q) = \begin{cases} 
\|B_a q\|_\infty, & \text{if } q_1 \leq 0 \text{ and } q_2 \leq 0, \\
\|B_b q\|_1, & \text{if } q_1 \leq 0 \text{ and } q_2 > 0, \\
\|B_c q\|_2, & \text{if } q_1 > 0 \text{ and } q_2 \leq 0, \\
\|B_d q\|_2, & \text{if } q_1 > 0 \text{ and } q_2 > 0,
\end{cases}$$

$$B_a = \begin{bmatrix} 1/2 & 0 \\
0 & 1 \end{bmatrix}, \quad B_b = \begin{bmatrix} 1/2 & 0 \\
0 & 1/2 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\
0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1 \\
0 & 1/2 \end{bmatrix}$$
Example: Multiple Vehicles

- Two robots free to move in the plane
  \[
  H(x, q) = G(x, q) - 1 = \left\| \left( f_\alpha \| q_1^\alpha, q_2^\alpha \|_2, f_\beta \| q_1^\beta, q_2^\beta \|_2 \right) \right\|_1 - 1
  \]
- No easy analytic solution for update from a single simplex
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    – Discretization
    – Causality
    – Algorithm
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Monotone Acceptance OUM (MAOUM)

• Like AFOUM
  – Modification of FMM to solve problems with general convex speed profiles in $O(N \log N)$
  – Semi-structured and unstructured simplicial grids

• Unlike AFOUM
  – Dijkstra-like algorithm: computes solution in order of nondecreasing value
  – Standard convergence proof [Barles & Souganidis, 1991]
  – Simple conversion to a Dial-like algorithm that sorts and accepts solution values using buckets
  – Stencil size adjusts to the local level of grid refinement
  – No accepted front
  – Initial pass through grid to generate stencils
  – Must store stencils
Semi-Lagrangian Discretization

- Same as AFOUM [Sethian & Vladimirsky, SINUM, 2003]
- Barycentric coordinates within stencil simplex

\[ \tilde{x}_s(\zeta) = \sum_{i=1}^{n_s} \zeta_i x_i^s \]

\[ \tau_s(x, \zeta) = \| \tilde{x}_s(\zeta) - x \| \]

\[ a_s(x, \zeta) = \frac{\tilde{x}_s(\zeta) - x}{\tau_s(x, \zeta)} \]

- Node value update:

\[ \bar{v}(x) = \min_{s \in S(x)} \min_{\zeta \in \Xi_{n_s}} \left\{ \frac{\tau_s(\zeta)}{f(x, a_s(\zeta))} + \sum_{i=1}^{n_s} \zeta_i \bar{v}(x_i^s) \right\} \]
Convergence

- **Update:** \( \mathcal{v}(x) = \min_{s \in \mathcal{S}(x)} \min_{\zeta \in \Xi_{ns}} \left\{ \frac{\tau_s(\zeta)}{f(x, a_s(\zeta))} + \sum_{i=1}^{ns} \zeta_i \mathcal{v}(x_i^s) \right\} \)

- **Consistency, stability, and monotonicity of discretization sufficient for convergence** [Barles & Souganidis, 1991]

- **Monotonicity**
  - Requires that if none of stencil node’s values decrease then the update value should not decrease
  - Follows from barycentric coordinates being nonnegative

- **Consistency**
  - Implied by *directional-completeness (DC):*
    \[ \bigcup_{s \in \mathcal{S}(x)} \{ a_s(\zeta) \mid \zeta \in \Xi_{ns} \} = \mathcal{A} \]
Causality

- Required for Dijkstra-like algorithm
- Causality: a node’s solution value is dependent only on smaller values in stencil
- Update from a single simplex:

\[
\nu_s(x) = \min_{\zeta \in \Xi_{n_s}} \left\{ \frac{\tau_s(\zeta)}{f(x, a_s(\zeta))} + \sum_{i=1}^{n_s} \zeta_i \nu(x_i^s) \right\}
\]

- Minimizer is \( \tilde{\zeta}^s \)
- Definition: the update equation is causal for node \( x \) and simplex \( s \) if

\[
\zeta_i^s > 0 \text{ implies } \nu_s(x) > \nu(x_i^s), \text{ for } 1 \leq i \leq n_s
\]
Negative-Gradient-Acuteness (NGA)

- Control-theoretic PDE:
  \[
  \max_{a \in \mathcal{A}} \left[ (-q \cdot a) f(x, a) \right] = (-q \cdot a_s(\hat{\zeta})) f(x, a_s(\hat{\zeta})) = 1
  \]

- Definition: simplex \( s \) is *negative-gradient-acute (NGA)* for node \( x \) if
  \[
  \text{for all } q \in \mathbb{R}^d \text{ and } \hat{\zeta} \in \mathbb{N}_s, \quad (x^s_i - x) \cdot (-q) > 0
  \]

- Theorem: NGA implies causality
Anisotropy-Angle-Boundedness (AAB)

- Local anisotropy:
  \[ \Upsilon(x) = \frac{\hat{f}(x)}{\tilde{f}(x)} \]

- Definition: simplex \( s \) is \textit{anisotropy-angle-bounded} (AAB) for node \( x \) if
  \[ \hat{\alpha}_s < \arcsin\left(\frac{1}{\Upsilon(x)}\right) \]

- Theorem: AAB implies NGA
Stencil Generation Algorithm

- Find a set of simplices that is DC and AAB
- Convergence:
  - DC sufficient for consistency
- Dijkstra-like algorithm
  - AAB sufficient for causality
- May generate stencil bigger than necessary
Experiment: Rectangular Speed Profile

- Homogeneous speed profile
- Boundary condition specified at origin
- Grid refined where solution and characteristics are highly curved
Results: Rectangular Speed Profile

- MAOUM and AFOUM on uniform and nonuniform grids
- Maximum and average error versus number of updates
- Nonuniform grid has better error convergence rate for both algorithms than nonuniform grid
- MAOUM on nonuniform grid has smallest error
Example: Seismology

- Example from [Sethian & Vladimirsky, SINUM, 2003]
- Compute first-arrival times for a wave propagating through a layered medium from source at origin
- Speed profile is an ellipse
  - Dimensions constant within layer
  - Long axis is aligned with tangent of sinusoid

![Graph showing wave propagation and speed profile](image_url)
Example: Robot Path Planning

- Robot wants to reach goal in minimal time avoiding obstacles and fighting a fierce wind
- Speed profile: $A_f(x) = \{ y \mid \| y - \vec{f}_w(x) \| \leq f_r \}$
- To compute optimal path:
  $$\frac{dx}{dt} = \vec{f}_w(x) - f_r \frac{Du(x)}{\| Du(x) \|} = f(x, \hat{a}(x)) \hat{a}(x)$$
Summary and Future Work

• Solution of static Hamilton-Jacobi PDEs can be applied to robotic path planning and seismology problems
• Dijkstra-like algorithms: computes solution in order of nondecreasing value in a single pass through grid nodes
• FMM for Axis-aligned HJ PDEs
  – FMM can solve many anisotropic problems on an orthogonal grid
  – Update can be made more efficient by eliminating potential neighbors and orthants
• MAOUM
  – Modification of FMM to solve problems with general convex speed profiles
  – Computes and stores causal stencil in an initial pass through grid
  – Simple conversion to a Dial-like algorithm that sorts and accepts solution values using buckets
  – Error-control adaptive-grid version
Relevant Publications

• [An ordered upwind method with precomputed stencil and monotone node acceptance for solving static convex Hamilton-Jacobi equations, Alton & Mitchell, in revision, JSC]
• [Fast marching methods for stationary Hamilton-Jacobi equations with axis-aligned anisotropy, Alton & Mitchell, SINUM, 2008]
• [Efficient dynamic programming for optimal multi-location robot rendezvous, Alton & Mitchell, CDC, 2008]
• [Optimal path planning under different norms in continuous state spaces, Alton & Mitchell, ICRA, 2006]