Monte Carlo Methods

(Estimators, On-policy/Off-policy Learning)

Julie Nutini

MLRG - Winter Term 2

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  - Episodic tasks ensure well-defined returns are available.
  - Incremental in an episode-by-episode sense.
    - Update value estimates/policies after completion of episode.

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- Both converge asymptotically.

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Initialize:

$\pi \leftarrow$ policy to be evaluated

$V \leftarrow$ an arbitrary state-value function

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Repeat forever:

(a) Generate an episode using π

(b) For each state s appearing in the episode:

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- Sequence of averages converges to expected value of $V^{\pi}(s)$.

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$\rightarrow$ Find state-value function for policy by MC approach.

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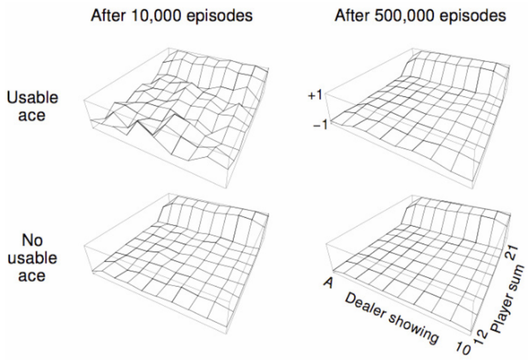

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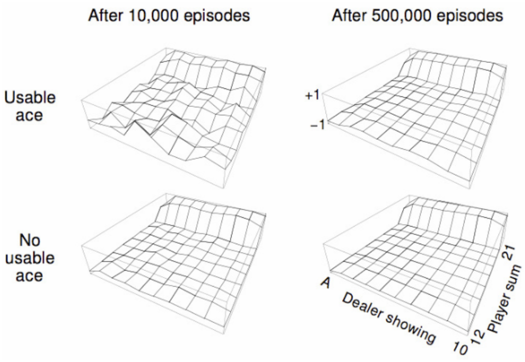

- Higher number of games (episodes), better approximation.
- Estimates for states with useable ace less certain.

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$\rightarrow$ Generating sample games easy.

- MC methods can be better, even when complete knowledge of environment's dynamics is known.

Backup Diagram for Monte Carlo

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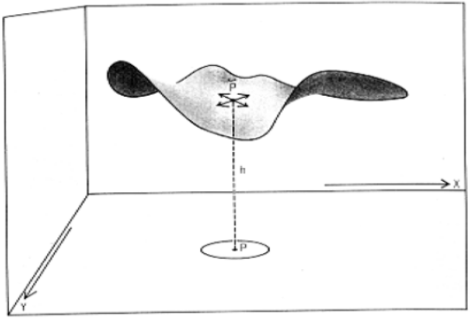
- Entire episode.
    - Rather than one-step transitions.
- Only one choice at each state.
    - DP explores all possible transitions.
- MC does not bootstrap.
    - Independent estimates for each state.
- Time required to estimate one state independent of total number of states.

The Power of Monte Carlo

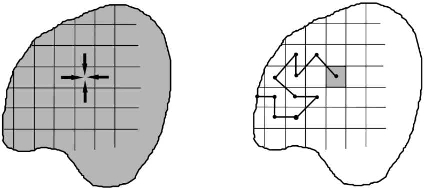
- E.g., elastic membrane (Dirichlet Problem)
  - How do we compute the shape of the surface?
    → Geometry of wire frame is known.



The Power of Monte Carlo

Relaxation

Kakutani's algorithm, 1945



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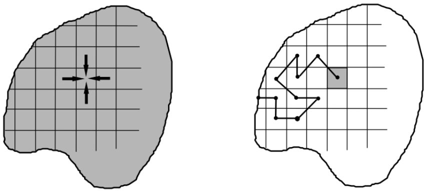

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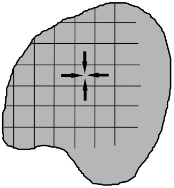

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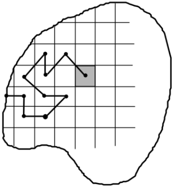

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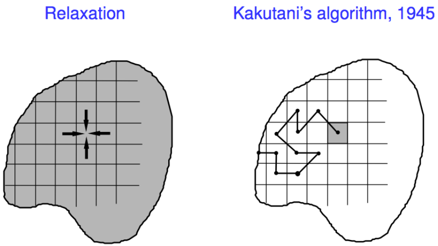

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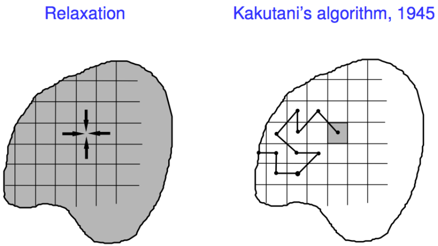

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$\rightarrow$ Local consistency.

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  - Without model, need to also estimate action values.
    - $\rightarrow$ We want to learn Q^*.
- Policy evaluation problem for action values:
  - Estimate $Q^\pi(s, a)$, the expected return starting from state s, taking action a, then following policy π.

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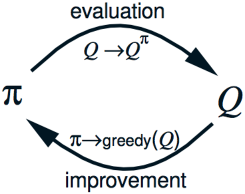
  - **Exploring starts**: Every state-action pair has non-zero probability of being starting pair.
  - **Alternative**: Only consider policies that are stochastic with nonzero probability of selecting all actions (later).

Monte Carlo Control

- Using MC estimation to approximate optimal policies.



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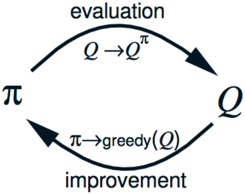

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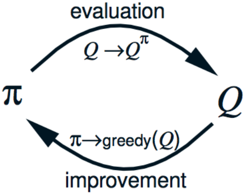

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- Policy improvement (I):
  - Greedify policy wrt current action-value function,

$$\pi(s) = \underset{a}{\text{argmax}}\, Q(s,a).$$

Convergence of MC Control

- Greedified policy meets conditions for policy improvement:

$$Q^{\pi_k}(s, \pi_{k+1}(s)) \quad = \quad Q^{\pi_k}(s, \operatorname*{argmax}_a Q^{\pi_k}(s, a))$$

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- To solve the latter:
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  - Alternate between evaluation & improvement per episode.

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Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s, a) \leftarrow$ arbitrary
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Repeat forever:
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    (b) For each pair s, a appearing in the episode:
        $R \leftarrow$ return following the first occurrence of s, a
        Append R to $Returns(s, a)$
        $Q(s, a) \leftarrow$ average($Returns(s, a)$)
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- Convergence to optimal fixed point seems inevitable.
- Open problem: Proving convergence to optimal fixed point.

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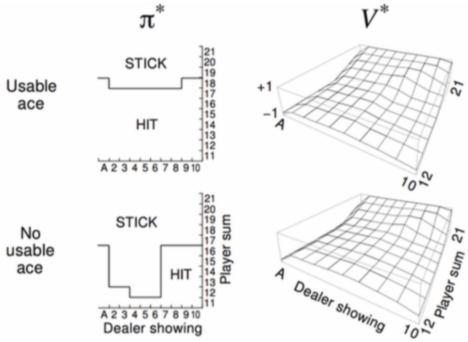

- Randomly select with equal prob. dealer's cards, player's sum and whether or not player has usable ace.

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- On-policy: Evaluate/improve policy while using for control.
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On-Policy Monte Carlo Control

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  - Need soft policies: $\pi(s,a) > 0$ for all $s \in \mathcal{S}$ and $a \in \mathcal{A}(s)$.
  - E.g., An ϵ-greedy policy is an example of ϵ-soft policy,

$$\pi(s,a) \geq \frac{\epsilon}{|\mathcal{A}(s)|}, \quad \forall\ s, a, \text{ and some } \epsilon > 0.$$

On-Policy MC Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \leftarrow$ arbitrary
  $Returns(s,a) \leftarrow$ empty list
  $\pi \leftarrow$ an arbitrary ε-soft policy

Repeat forever:
  (a) Generate an episode using π
  (b) For each pair s, a appearing in the episode:
    $R \leftarrow$ return following the first occurrence of s, a
    Append R to $Returns(s,a)$
    $Q(s,a) \leftarrow \text{average}(Returns(s,a))$
  (c) For each s in the episode:
    $a^* \leftarrow \arg\max_a Q(s,a)$
    For all $a \in \mathcal{A}(s)$:

$$\pi(s,a) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(s)| & \text{if } a = a^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq a^* \end{cases}$$

- Encourages exploration of nongreedy actions.

Learning About π While Following π'

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- We have n_s returns, $R_i(s)$, from state s, with:
  - probability $p_i(s)$ of being generated by π
  - probability $p'_i(s)$ of being generated by π'
- Estimate using weighted importance sampling:

$$V_\pi(s) \approx \frac{\sum_{i=1}^{n_s} \frac{p_i(s)}{p'_i(s)} R_i(s)}{\sum_{i=1}^{n_s} \frac{p_i(s)}{p'_i(s)}}$$

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- Depends on the environmental probabilities $p_i(s)$ and $p_i'(s)$.
  - Normally considered unknown in MC applications.

Learning About π While Following π'

- However,

$$p_i(s_t) = \prod_{k=t}^{T_i(s)-1} \pi(s_k, a_k)\mathcal{P}_{s_k s_{k+1}}^{a_k}$$

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$\rightarrow$ The weights only depend on the two policies!

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    - Estimation policy: evaluated and improved.
        - Deterministic, greedy.
    - Two policies may be unrelated.

Off-Policy MC Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \leftarrow$ arbitrary
  $N(s,a) \leftarrow 0$; Numerator and
  $D(s,a) \leftarrow 0$; Denominator of $Q(s,a)$
  $\pi \leftarrow$ an arbitrary deterministic policy

Repeat forever:
  (a) Select a policy π' and use it to generate an episode:
    $s_0, a_0, r_1, s_1, a_1, r_2, \ldots, s_{T-1}, a_{T-1}, r_T, s_T$
  (b) $\tau \leftarrow$ latest time at which $a_\tau \neq \pi(s_\tau)$
  (c) For each pair s, a appearing in the episode at time τ or later:
    $t \leftarrow$ the time of first occurrence of s, a such that $t \geq \tau$
    $w \leftarrow \prod_{k=t+1}^{T-1} \frac{1}{\pi'(s_k, a_k)}$
    $N(s,a) \leftarrow N(s,a) + wR_t$
    $D(s,a) \leftarrow D(s,a) + w$
    $Q(s,a) \leftarrow \frac{N(s,a)}{D(s,a)}$
  (d) For each $s \in \mathcal{S}$:
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- Method learns only from tails of episodes.
  - Potentially cause slow learning.

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  - Dealer is showing 2.
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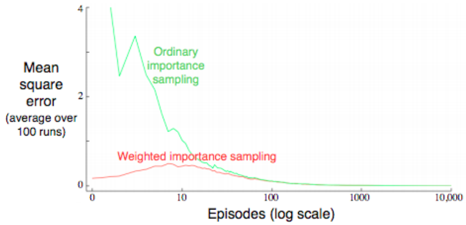

- Optimal value of state under target policy $\approx$ -0.27726.

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  - **No need** to learn about ALL states.
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- Control methods and approximating action-value functions.
  - MC intermix policy evaluation and policy improvement.
- One issue to watch for: maintaining sufficient exploration.
  - Exploring starts.
  - On-policy and off-policy methods.