Is Greedy Coordinate Descent a Terrible Algorithm?

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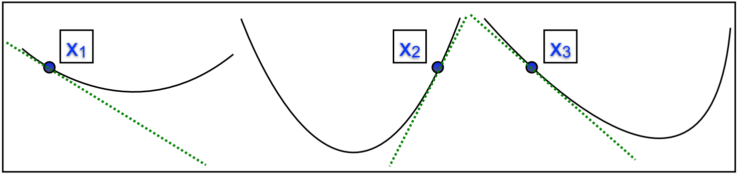

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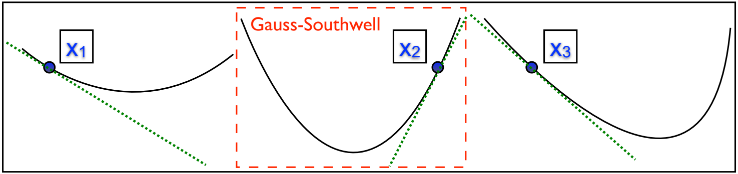

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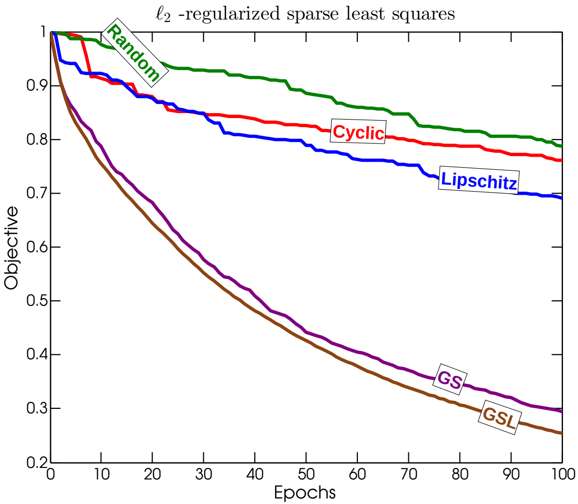

- All rules have similar costs for this problem.

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→ E.g., lattice-structured graphs and complete graphs.

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- If f is twice-differentiable, equivalent to

$$\nabla^2_{ii} f(x) \leq L, \qquad \nabla^2 f(x) \succeq \mu \mathbb{I}.$$

Randomized Coordinate Descent

Coordinate descent with constant step-size $\frac{1}{L}$ update:

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- Since $Ln \geq L_f \geq L$, coordinate descent is slower *per iteration*, but n coordinate iterations are faster than one gradient iteration.

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Using $\|\nabla f(x^k)\|^2 \leq n\|\nabla f(x^k)\|_\infty^2$ we get

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Lipschitz Sampling

Consider the case where we have an L_i for each coordinate

$$|\nabla_i f(x + \alpha e_i) - \nabla_i f(x)| \leq L_i|\alpha|,$$

and we use a coordinate-dependent step-size,

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- The answer is neither!

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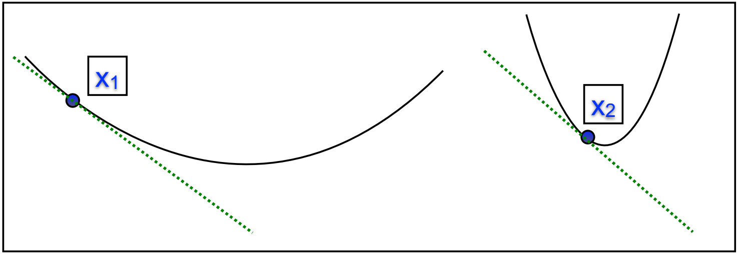

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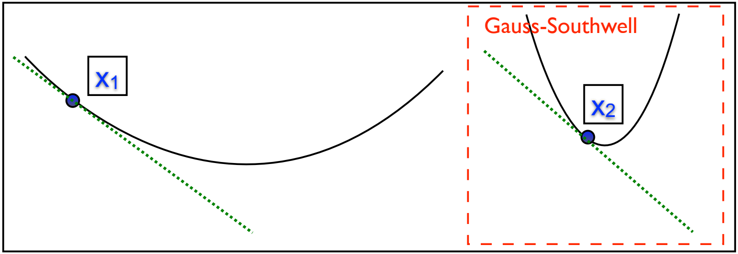

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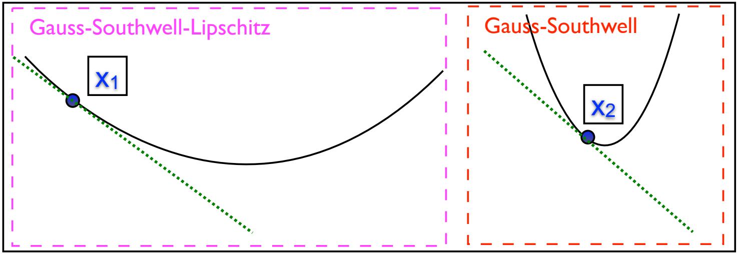

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- Gives tighter bound for maximum improvement rule.

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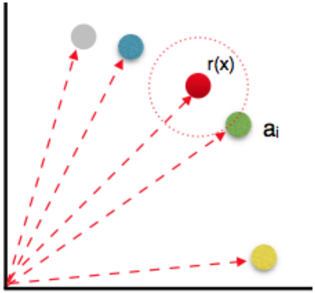

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- Approximation is exact if $\|a_i\| = 1$ for all i.

Gauss-Southwell-Lipschitz as Nearest Neighbour

Dhillon et al. [2011] approximate GS as nearest neighbour,

$$\operatorname*{argmin}_i \|r(x^k) - a_i\| = \operatorname*{argmax}_i \left\{ |\nabla_i f(x^k)| - \frac{1}{2}\|a_i\|^2 \right\}.$$

- Approximation is exact if $\|a_i\| = 1$ for all i.

Usually $L_i = \gamma\|a_i\|^2$, in this case exact GSL is a nearest neighbour problem,

$$\operatorname*{argmin}_i \left\| r(x^k) - \frac{a_i}{\|a_i\|} \right\| = \operatorname*{argmin}_i \left\{ \frac{|\nabla_i f(x^k)|}{\sqrt{L_i}} \right\}.$$

- See paper and poster for numerical results on the nearest neighbour.

Proximal Coordinate Descent

Consider the following problem

$$\min_{x \in \mathbb{R}^n} F(x) \equiv f(x) + \sum_i g_i(x_i),$$

where f is smooth and g_i might be non-smooth.

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Apply proximal-gradient style update,

$$x^{k+1} = \textbf{prox}_{\frac{1}{L} g_{i_k}} \left[x^k - \frac{1}{L} \nabla_{i_k} f(x^k) e_{i_k} \right],$$

where

$$\textbf{prox}_{\alpha g}[y] = \operatorname*{argmin}_{x \in \mathbb{R}^n} \frac{1}{2}\|x - y\|^2 + \alpha g(x).$$

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- GS-q: Maximize progress under quadratic approximation of f,

$$i_k = \operatorname*{argmin}_i \left\{ \min_d f(x^k) + \nabla_i f(x^k) d + \frac{Ld^2}{2} + g_i(x_i^k + d) - g_i(x_i^k) \right\}.$$

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$\rightarrow$ If you use L_i in the GS-q rule, it is a generalization of GSL rule.

Proximal Gauss-Southwell Convergence Rate

For random selection, Richtárik and Takáč [2014] show

$$\mathbb{E}[F(x^{k+1})] - F(x^k) \leq \left(1 - \frac{\mu}{Ln}\right)[F(x^k) - F(x^*)].$$

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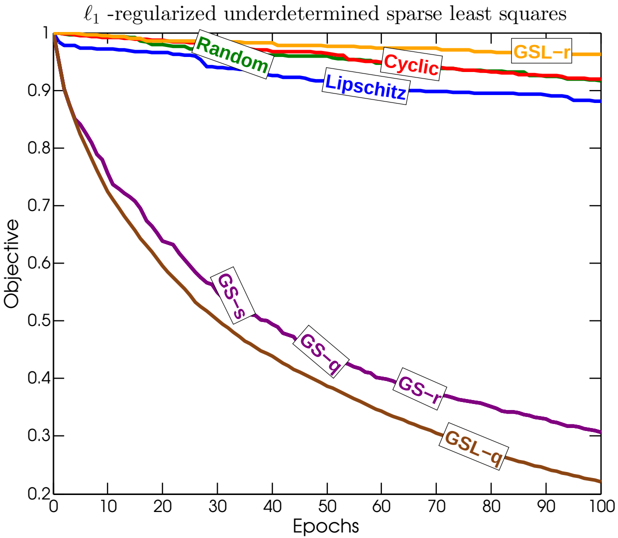
- But, again theory disagrees with practice...

Comparison of Proximal Gauss-Southwell Rules



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- Current/future work:
  - accelerated/parallel methods [Fercocq & Richtárik, 2013]
  - primal-dual methods [Shalev-Schwartz & Zhang, 2013]
  - without strong-convexity [Luo & Tseng, 1993]

Thank you!