Marks

[8] **1.** Consider the problem: maximize x_1 subject to $x_1 \geq 6$, $x_1 \leq 10$, $x_1 \geq 7$, $x_1 \geq 0$. Write this as a linear program in standard form. Use the two-phase method, **adding an auxilliary variable x_0 to EVERY slack variable equation in the dictionary,** to solve this LP. Use the smallest subscript rule to break any ties for entering or leaving variables.

Answer: In standard form this is

$$\max x_1, \quad \text{subject to}$$
$$-x_1 \leq -6$$
$$x_1 \leq 10$$
$$-x_1 \leq -7$$
$$x_1 \geq 0$$

This gives rise to an initial dictionary:

$$z = x_1$$
$$x_2 = -6 + x_1$$
$$x_3 = 10 - x_1$$
$$x_4 = -7 + x_1$$

Due to the negative constants in the constraints, we use the two phase method, beginning by adding x_0 as non-basic and modifying the objective:

$$w = -x_0$$
$$x_2 = -6 + x_1 + x_0$$
$$x_3 = 10 - x_1 + x_0$$
$$x_4 = -7 + x_1 + x_0$$

We have x_0 enter and choose the leaving variable so that the dictionary becomes feasible, namely x_4 leaves:

$$x_0 = 7 - x_1 + x_4$$
$$x_2 = 1 + x_4$$
$$x_3 = 17 - 2x_1 + x_4$$
$$w = -7 + x_1 - x_4$$

So x_1 enters and x_0 leaves (and so we omit w):

$$x_1 = 7 - x_0 + x_4$$
$$x_2 = 1 + x_4$$
$$x_3 = 3 + 2x_0 - x_4$$

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This gives us a feasible dictionary with x_0 non-basic. So we eliminate the x_0 to get the feasible dictionary (and use our original objective $z = x_1$), to start the second phase:

$$\begin{aligned} x_1 &= 7 + x_4 \\ x_2 &= 1 + x_4 \\ x_3 &= 3 - x_4 \\ z = x_1 &= 7 + x_4 \end{aligned}$$

So x_4 enters and x_3 leaves:

$$\begin{aligned} x_4 &= 3 - x_3 \\ x_1 &= 10 - x_3 \\ x_2 &= 4 - x_3 \\ z &= 10 - x_3 \end{aligned}$$

Since all the coefficients of the z row are zero or negative, we are done and the optimal solution is $x_1 = 10$ with $z = 10$.

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[12] **2.** (4 points for each part) Let

$$A_1 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix}$$

(a) Find the value of "Alice announces a pure strategy," "Betty announces a pure strategy," and the value of the mixed strategy games for the matrix game A_1; **show your work and justify your conclusions.**

Answer: If Alice announces row 1, Betty chooses the minimum of 1 and 2, namely 2; if Alice announces row 2, Betty chooses 3; so the value of "Alice announces a pure strategy" is 3 (and Alice announces row 2). If Betty announces column 1, Alice chooses 4, and if Betty announces column 2, Alice chooses 3; so the value of "Betty announces a pure strategy" is 3 (and Betty announces column 2). Since these two values are both 3, the mixed strategy games have the value 3.

(b) Same question as (a) for A_2.

Answer: By similar reasoning, the value of "Alice announces a pure strategy" is -1 (and Alice announces row 1), and the value of "Betty announces a pure strategy" is 2 (and Betty announces column 2). Since these values are different, the optimal mixed strategy for Alice is balanced, and the value, v, is given by solving

$$[x_1 \ \ 1 - x_1] \begin{bmatrix} -1 & 2 \\ 4 & -3 \end{bmatrix} = [v \ \ v],$$

which gives

$$(-1)x_1 + 4(1 - x_1) = v = 2x_1 + (-3)(1 - x_1)$$

so $x_1 = 7/10$ and $v = 1/2$.

(c) Let A be a 100×2 matrix game. Argue that Alice has an optimal mixed strategy where she plays at most two rows.

Answer: By adding a matrix of constant entries to A we may assume that all of A's entries are non-negative, and so the value of the mixed strategy games, v, is non-negative. The linear program for Alice's mixed strategy is to maximize v subject to

$$1 - x_1 - \ldots - x_{99} \geq 0,$$

$$[x_1 \ x_2 \ \ldots x_{99} \ 1 - x_1 - \cdots - x_{99}]A \geq [v \ \ v]$$

and $x_1, \ldots, x_{99}, v \geq 0$. We know this is a feasible linear program with v bounded by the largest entry of A. So when we run the simplex method we will have a final dictionary with 3 basic variables, one of which must be v since we know $v > 0$; this leaves two other basic variables, so at most two of $x_1, \ldots, x_{100}$ can be nonzero in the optimal solution, with $x_{100} = 1 - x_1 - \cdots - x_{99}$ corresponding to how Alice plays the last row. Hence at least 98 rows (i.e., strategies) of Alice's optimal mixed strategy are not played in this optimal solution.

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[8] **3.** What is the value of the mixed strategy games for the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 9 & 10 & 11 & 12 \\ 10 & 9 & 12 & 11 \end{bmatrix} ?$$

Justify your reasoning and any calculations you perform.

Answer: All values in the first two rows are smaller than those in the last two rows, so the last two rows dominate the first two; i.e., Alice can discard rows 1 and 2. This leaves the matrix game:

$$A = \begin{bmatrix} 9 & 10 & 11 & 12 \\ 10 & 9 & 12 & 11 \end{bmatrix},$$

whereupon columns 1 and 2 dominate columns 3 and 4 (i.e., Betty will never play columns 3 or 4). This leaves the matrix game

$$A = \begin{bmatrix} 9 & 10 \\ 10 & 9 \end{bmatrix},$$

where the values of "announcing a pure strategy" are 9 and 10, and hence the value of the mixed strategy games are given by

$$[x_1 \quad 1 - x_1] \begin{bmatrix} 9 & 10 \\ 10 & 9 \end{bmatrix} = [v \quad v],$$

which yields $x_1 = 1/2$ and $v = 19/2$.

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[8] **4.** Consider the matrix game:

$$A = \begin{bmatrix} 5 & 4 & 3 \\ 6 & 1 & 4 \\ 2 & 5 & 5 \end{bmatrix}.$$

Carol sets up the linear program: maximize v subject to $1 - x_1 - x_2 \geq 0$ and

$$[x_1 \quad x_2 \quad 1 - x_1 - x_2] \begin{bmatrix} 5 & 4 & 3 \\ 6 & 1 & 4 \\ 2 & 5 & 5 \end{bmatrix} \geq [v \quad v \quad v]$$

and $x_1, x_2, x_3, v \geq 0$. Carol tells you that she arrived at the dictionary

$$\begin{aligned} z =& (79/20) - (7/20)x_4 - (1/4)x_5 - (2/5)x_6 \\ v =& (79/20) - (7/20)x_4 - (1/4)x_5 - (2/5)x_6 \\ x_1 =& (9/20) + (3/20)x_4 + (1/4)x_5 - (2/5)x_6 \\ x_2 =& (3/20) + (1/20)x_4 - (1/4)x_5 + (1/5)x_6 \\ x_3 =& (2/5) - (1/5)x_4 - (0)x_5 + (1/5)x_6 \end{aligned}$$

(a) What is the value of the game "Alice announces a mixed strategy" based on this final dictionary? What is Alice's optimum mixed strategy, based on this final dictionary?

Answer: From the v, x_1, x_2, x_3 values in the final dictionary, the value is $79/20$ and Alice plays rows 1,2,3 the respective fractions $9/20$, $3/20$, and $2/5$.

(b) Dina does not believe that the above final dictionary is correct. Show that your answers in (a) are correct with an argument that doesn't rely on the final dictionary. [You may use the information in the dictionary, but your argument cannot rely on this dictionary being 100% correct.]

Answer: There are a number of different methods; we shall present a few. All the methods here make use of the fact that From the strategy of Alice, we compute

$$[9/20 \ 3/20 \ 8/20] \begin{bmatrix} 5 & 4 & 3 \\ 6 & 1 & 4 \\ 2 & 5 & 5 \end{bmatrix} = [79/20 \ 79/20 \ 79/20]$$

and it follows that the value of the mixed stategry games is at least $79/20$.

[A number of people claimed that the above was enough to prove that the value of the mixed strategy games was equal to $79/20$. Even for 2×2 games you need more information; and for 3×3 games or larger these matters are much more subtle.]

Method 1: From the homework, we know that (if the dictionary is correct) Betty's strategy should be the coefficients of the z row, namely $7/20, 1/4, 2/5$.

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Regardless of whether or not the dictionary is correct, we can see what happens when Betty plays this strategy: so we calculate

$$\begin{bmatrix} 5 & 4 & 3 \\ 6 & 1 & 4 \\ 2 & 5 & 5 \end{bmatrix} \begin{bmatrix} 7/20 \\ 1/4 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 79/20 \\ 79/20 \\ 79/20 \end{bmatrix}.$$

It follows that the mixed value is no more than $79/20$. Hence it equals $79/20$.

Method 2: The z row of the final dictionary always provides a proof that the objective is optimal. Specifically we should take the inequality corresponding to x_4, multiply by $7/20$, do similarly with x_5 and x_6, and add: so we take

$$(5x_1 + 6x_2 + (1 - x_1 - x_2)2 \geq v) \quad \textbf{both sides times } 7/20$$

$$(4x_1 + 1x_2 + (1 - x_1 - x_2)5 \geq v) \quad \textbf{both sides times } 1/4$$

$$(3x_1 + 4x_2 + (1 - x_1 - x_2)5 \geq v) \quad \textbf{both sides times } 2/5$$

and add them, which gives

$$79/20 + 0x_1 + 0x_2 \geq v,$$

and hence v cannot be greater than $79/20$.

Method 3: We can verify that the final row

$$z = (79/20) - (7/20)x_4 - (1/4)x_5 - (2/5)x_6$$

is correct by substituting in the expressions for the slack variables x_4, x_5, x_6 given by the linear program (i.e., $x_4 = 5x_1 + 6x_2 + (1 - x_1 - x_2)2 - v$, etc.). Once we have verified this line, we know that z cannot be larger than $79/20$.

[Methods 2 and 3 are very similar in spirit.]

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Be sure that this examination has 6 pages including this cover

The University of British Columbia

Midterm Examinations - October 2015

Mathematics 340

Closed book examination | Time: 45 minutes

Name ______________________ **Signature** ______________________

Student Number ______________________ **Instructor's Name** ______________________

Section Number ______________________

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