HOMEWORK 6 SOLUTIONS, MATH 340, FALL 2015

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(1) The dictionary consists of

$$\begin{aligned} \vec{x}_B = &\quad A_B^{-1}\vec{b} \quad -A_B^{-1}A_N\vec{x}_N \\ z = &\quad \vec{c}_B^{\mathrm{T}}A_B^{-1}\vec{b} \quad +(\vec{c}_N^{\mathrm{T}} - \vec{c}_B^{\mathrm{T}}A_B^{-1}A_N)\vec{x}_N \end{aligned}$$

Since there are always m basic variables, and A_B is $m \times m$, naively inverting A_B will take order m^3 FLOPs. Multiplying $\vec{c}_B^{\mathrm{T}}A_B^{-1}$ will take order m^2 FLOPs. Also the cost of computing $A_B^{-1}A_N\vec{x}$ is mostly not needed: we only compute A_B^{-1} times the A_N row corresponding to the entering variable, which again costs order m^2 FLOPs. The only operation that can be expensive is multiplying $\vec{c}_B^{\mathrm{T}}A_B^{-1}$ by A_N to compute the z row.

If A_N is sparse, then although $\vec{c}_B^{\mathrm{T}}A_B^{-1}$ is m dimensional and A_N is $m \times n$, the fact that A_N is mostly zeros means that will only need a number of multiplications and additions each equal to the number of nonzero entries of A_N. Hence this step will require mn times 5%; an ordinary pivot (at least when the dictionary is dense) requires mn additions and mn multiplications. Hence the revised simplex method requires order m^3 plus $2mn/20$ FLOPs (one addition and one multiplication for each of mn times 5% entries of A_N), as opposed to $2mn$ for a standard simplex method pivot. So you save roughly a factor of 20 for each pivot (neglecting the order m^3 terms, since $mn = m^5$ is much larger).

(2) One again, the standard simplex method requires roughly $2mn$ operations to update each dictionary. And once again, the revised simplex method takes order m^3 operations plus the cost of taking the product of $\vec{c}_B^{\mathrm{T}}A_B^{-1}$ with A_N. However, rather than multiply all of A_N you only have to do the columns of A_N corresponding to the non-basic variables that might enter. So if you can identify $n/10$ variables from which to choose, you only need to compute $\vec{c}_B^{\mathrm{T}}A_B^{-1}$ times $n/3$ of the n columns of A_N. Hence you save a factor of 3 for each pivot where this is possible. In general, if $n/3$ is replaced with n', then you save a factor of n/n'; so for $n' = n/30$ you save a factor of 30, and for $n' = \sqrt{n}$ you save a factor of $\sqrt{n}$.

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