HOMEWORK 5 SOLUTIONS, MATH 340, FALL 2015

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(1) Consider the constraint $x_1+x_2 \le 1$: for any feasible $\vec{x}$ such that $x_1+x_2 < 1$, it is always possible to increase x_1 (or x_2) a bit and preserve the inequality $x_1 + x_2 \le 1$. However, since the entries of A are all positive, any increase in x_1 will strictly increase v. Hence $\vec{x}$ cannot be an optimal solution for the objective $z = v$ unless $x_1 + x_2 = 1$.

(2) The dual LP is

$$\begin{array}{llllll} \text{minimize} & w, & \text{subject to} & & \\ & 11y_1 & +9y_2 & \le & w \\ & 8y_1 & +12y_2 & \le & w \\ & y_1 & +y_2 & \ge & 1 \\ \text{and} & & y_1, y_2, w \ge 0. & & \end{array}$$

(We have chosen to call the third dual variable w instead of y_3 because the minimum value of w will be the value that will give an upper bound on the objective, z.) Let y_3, y_4, y_5 be the slack variables for the first three inequalities,

$$y_3 = w - 11y_1 - 9y_2, \quad y_4 = w - 8y_1 - 12y_2, \quad y_5 = -1 + y_1 + y_2,$$

and let x_3, x_4, x_5 be the slack variables for the primal (as given by the solutions to Homework 3):

$$x_3 = -v + 11x_1 + 8x_2, \quad x_4 = -v + 9x_1 + 12x_1, \quad x_5 = 1 - x_1 - x_2.$$

We have the correspondence:

$$x_1 \leftrightarrow y_3, \quad x_2 \leftrightarrow y_4, \quad v \leftrightarrow y_5, \quad x_3 \leftrightarrow y_1, \quad x_4 \leftrightarrow y_2, \quad x_5 \leftrightarrow w.$$

(a) $x_1 = x_2 = 1/2$, $v = 19/2$. From the solutions to Homework 3 we compute $x_3 = 0$, $x_4 = 1$, $x_5 = 0$. So x_1, x_2, x_4 are positive, which forces $y_3 = y_4 = y_5 = y_2 = 0$. $y_3 = y_4 = y_5 = 0$ give the three equations:

$$11y_1 = w, \quad 8y_1 = w, \quad y_1 = 1$$

(since $y_2 = 0$). This system is inconsistent, since $y_1 = 1$ forces both $w = 11$ and $w = 8$. Hence the proposed solution is not optimal.

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(b) $x_1 = 1/3$, $x_2 = 2/3$, $v = 9$; we have $x_3 = 0$, $x_4 = 2$, $x_5 = 0$. This forces $y_3 = y_4 = y_5 = y_2 = 0$, which we have seen in part (a) leads to an inconsistent system. Hence the proposed solution is not optimal.

(c) $x_1 = 2/3$, $x_2 = 1/3$, $v = 10$; we have $x_3 = 0$, $x_4 = 0$, $x_5 = 0$, which forces $y_3 = y_4 = y_5 = 0$. This gives the equations

$$0 = w - 11y_1 - 9y_2 = w - 8y_1 - 12y_2 = -1 + y_1 + y_2,$$

which we solve to find $y_1 = y_2 = 1/2$, $w = 10$, which is therefore an optimal solution (all the x's and y's are non-negative and satisfy complementary slackness).

(d) $x_1 = 1$, $x_2 = 0$, $v = 9$; we have $x_3 = 2$, $x_4 = 0$, $x_5 = 0$. This forces $y_3 = y_5 = y_1 = 0$. $y_3 = y_5 = 0$ gives the two equations

$$9y_2 = w, \quad y_2 = 1$$

(since $y_1 = 0$). This implies $y_2 = 1$, $w = 9$; the last variable to check is y_4, but

$$y_4 = w - 8y_1 - 12y_2 - v = 9 + 0 - 12 = -3,$$

which is negative. Hence the proposed solution is not optimal.

(e) $x_1 = 0$, $x_2 = 0$, $v = 0$; we have $x_3 = 0$, $x_4 = 0$, $x_5 = 1$. This forces $w = 0$. This leaves us with

$$y_3 = -11y_1 - 9y_2, \quad y_4 = -8y_1 - 12y_2, \quad y_5 = -1 + y_1 + y_2.$$

This set of three equations in five unknowns has many solutions, but it is not hard to see that none of these solutions can have i$y_1, \ldots, y_5$ all non-negative. Indeed, the equation $y_3 = -11y_1 - 9y_2$ cannot have either y_1 or y_2 positive, or else y_3 would be negative; so $y_1 = y_2 = y_3 = 0$. The equation $y_5 = -1 + y_1 + y_2$ then forces $y_5 = -1$, which is negative. Hence the proposed solution is not optimal.

(3) The dual LP is already written above. If Betty plays columns 1 and 2 with frequencies y_1 and y_2, then Alice chooses

$$\max(11y_1 + 9y_2, 8y_1 + 12y_2),$$

which is the same thing as the smallest v such that

$$11y_1 + 9y_2 \leq v \quad \text{and} \quad 8y_1 + 12y_2 \leq v.$$

Hence Betty will choose the non-negative y_1, y_2 subject to $y_1 + y_2 = 1$ such that v is minimized. There is no harm in replacing $y_1 + y_2 = 1$ with $y_1 + y_2 \geq 1$ here, since if $y_1 + y_2 > 1$, then one cannot be at optimality since Betty can choose a slightly smaller y_1 (or y_2) and get a smaller w value. Hence the dual LP above is precisely the LP that describes Betty's best mixed strategy, with w being the value of this game.

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