

HOMEWORK 4 SOLUTIONS, MATH 340, FALL 2015

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(1) (a)

$$\begin{array}{rcl} x_3 & = & 2 + \epsilon_1 \quad -x_1 \\ x_4 & = & 3 + \epsilon_2 \quad \quad -x_2 \\ x_5 & = & 5 + \epsilon_3 \quad -x_1 \quad -x_2 \\ \hline z & = & \quad \quad 2x_1 \quad +x_2 \end{array}$$

x_2 enters and x_4 leaves.

$$\begin{array}{rcl} x_3 & = & 2 + \epsilon_1 \quad -x_1 \\ x_2 & = & 3 + \epsilon_2 \quad \quad -x_4 \\ x_5 & = & 2 - \epsilon_2 + \epsilon_3 \quad -x_1 \quad +x_4 \\ \hline z & = & 3 + \epsilon_2 \quad +2x_1 \quad -x_4 \end{array}$$

x_1 enters; since $2 + \epsilon_1 > 2 - \epsilon_2 + \epsilon_3$ (we'd have a tie if the ϵ_i 's weren't there), x_5 leaves:

$$\begin{array}{rcl} x_3 & = & \epsilon_1 + \epsilon_2 - \epsilon_3 \quad +x_5 \quad -x_4 \\ x_2 & = & 3 + \epsilon_2 \quad \quad -x_4 \\ x_1 & = & 2 - \epsilon_2 + \epsilon_3 \quad -x_5 \quad +x_4 \\ \hline z & = & 7 - \epsilon_2 + 2\epsilon_3 \quad -2x_5 \quad +x_4 \end{array}$$

Now x_4 enters and x_3 leaves (this would be a degenerate pivot if not for the ϵ_i 's).

$$\begin{array}{rcl} x_4 & = & \epsilon_1 + \epsilon_2 - \epsilon_3 \quad +x_5 \quad -x_3 \\ x_2 & = & 3 - \epsilon_1 + \epsilon_3 \quad -x_5 \quad +x_3 \\ x_1 & = & 2 + \epsilon_1 \quad \quad -x_3 \\ \hline z & = & 7 + \epsilon_1 + \epsilon_3 \quad -x_5 \quad -x_3 \end{array}$$

This dictionary is final, and so the optimal solution is $x_1 = 2$, $x_2 = 3$. A picture is given Figure 1.

(b) The dictionaries look the same, with ϵ_1 and ϵ_3 interchanged. So the first dictionary is

$$\begin{array}{rcl} x_3 & = & 2 + \epsilon_3 \quad -x_1 \\ x_4 & = & 3 + \epsilon_2 \quad \quad -x_2 \\ x_5 & = & 5 + \epsilon_1 \quad -x_1 \quad -x_2 \\ \hline z & = & \quad \quad 2x_1 \quad +x_2 \end{array}$$

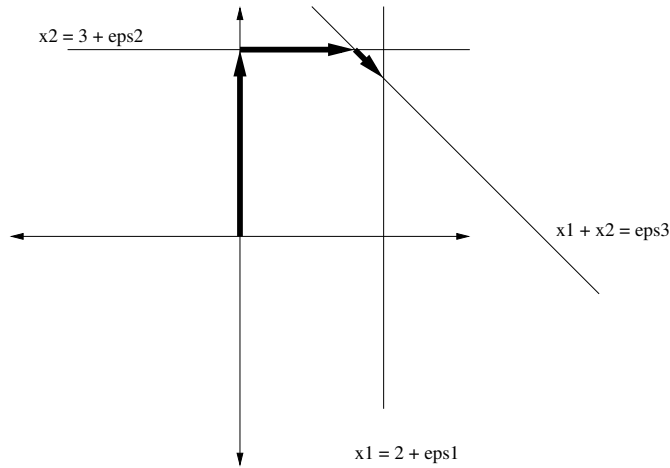
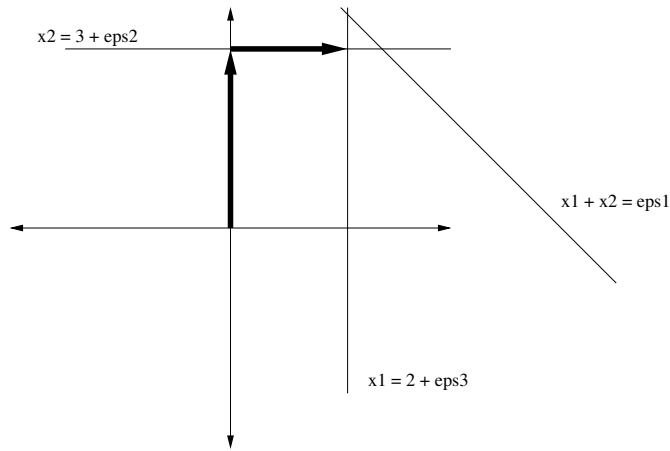


FIGURE 1. Drawing for the first simplex method

FIGURE 2. Drawing for the second simplex method, i.e., ϵ_1 and ϵ_3 interchanged

and the first pivot is x_2 enters, x_4 leaves, as before. But on the second pivot x_1 enters and x_3 leaves, since the the second dictionary looks like:

$$\begin{array}{rcl} x_3 & = & 2 + \epsilon_3 \quad -x_1 \\ x_2 & = & 3 + \epsilon_2 \quad \quad -x_4 \\ x_5 & = & 2 - \epsilon_2 + \epsilon_1 \quad -x_1 \quad +x_4 \\ \hline z & = & 3 + \epsilon_2 \quad +2x_1 \quad -x_4 \end{array}$$

and $2 + \epsilon_3 < 2 - \epsilon_2 + \epsilon_1$. Thus we get to the final dictionary

$$\begin{array}{rcl} x_5 & = & -\epsilon_3 - \epsilon_2 + \epsilon_1 \quad +x_3 \quad +x_4 \\ x_2 & = & 3 + \epsilon_2 \quad \quad -x_4 \\ x_1 & = & 2 + \epsilon_3 \quad -x_3 \\ \hline z & = & 7 + 2\epsilon_3 + \epsilon_2 \quad -2x_3 \quad -x_4 \end{array}$$

A picture is give in Figure 2.

- (c) Note that this final dictionary is different, with different basic variables! Also, we took one fewer pivots this new way.

Note also that this shows there is more than one final dictionary here. For the future, this means that there is more than one optimal solution for the dual problem; this is due to the three lines coinciding at the optimal solution for the primal LP.

When we perturb a degenerate problem, we can expect more than one geometry to emerge depending on the perturbation. The perturbed problems and the ϵ_i 's are guides to which pivots to take, so different geometries give rise to different sequences of dictionaries.

There are many other comments one could make along these lines (this is not a threat).

- (2) Here we are going to assume, for simplicity that $a, b > 0$ in optimality (we can always at $A + Bt_i$ to each y_i where A, B are large positive numbers to achieve this). The linear program is to minimize $M_1 + \dots + M_n$ subject to

$$-M_1 \leq y_1 - a - bt_1 \leq M_1, \quad -M_2 \leq y_2 - a - bt_2 \leq M_2, \quad \dots$$

We therefore have $n + 2$ decision variables, and $2n$ slack variables. When some $M_i = 0$, we have that both corresponding slack variables are zero; when some $M_i > 0$, only (and exactly) one slack variable is zero. Since $a, b > 0$ in optimality, and $n + 2$ variables are non-basic, there must be (at least) two of the M_i that are zero in the final dictionary. Hence we can say that there is always a solution, a, b , for which $y = a + bt$ goes through (at least) two of the data points.

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