HOMEWORK 2 SOLUTIONS, MATH 340, FALL 2015

JOEL FRIEDMAN

Problem 1(a)

We have the initial dictionary:

$$\begin{array}{rclrrr} z & = & & 6x_1 & +7x_2 \\ \hline x_3 & = & 5 & -x_1 & \\ x_4 & = & 8 & & -x_2 \\ x_5 & = & 10 & -x_1 & -x_2 \end{array}$$

If x_2 enters, then the most restrictive inequality is $x_4 \geq 0$ (since this imposes $x_2 \leq 8$, as opposed to $x_3 \geq 0$ which imposes no condition on x_2, and $x_5 \geq 0$ which imposes $x_2 \leq 10$). Hence x_2 enters and x_4 leaves the basis, so we pivot on

$$x_4 = 8 - x_2 \quad \text{which becomes} \quad x_2 = 8 - x_4$$

giving us the second dictionary

$$\begin{array}{rclrr} z & = & 56 & +6x_1 & -7x_4 \\ \hline x_3 & = & 5 & -x_1 & \\ x_2 & = & 8 & & -x_4 \\ x_5 & = & 2 & -x_1 & +x_4 \end{array}$$

Now x_1 must enter, and x_5 reaches zero first (at $x_1 = 2$) and leaves the basis, so we pivot on

$$x_5 = 2 - x_1 + x_4 \quad \text{which becomes} \quad x_1 = 2 - x_5 + x_4$$

giving us the third dictionary

$$\begin{array}{rclrr} z & = & 68 & -6x_5 & -x_4 \\ \hline x_3 & = & 3 & +x_5 & -x_4 \\ x_2 & = & 8 & & -x_4 \\ x_1 & = & 2 & -x_5 & +x_4 \end{array}$$

Since all the z row coefficients are negative, this gives us the optimal value $z = 68$ for $x_1 = 2$ and $x_2 = 8$ (in terms of the decision or original variables).

Problem 1(b)

Research supported in part by an NSERC grant.

We have the same initial dictionary:

$$\begin{array}{ccrrr} z & = & & 6x_1 & +7x_2 \\ \hline x_3 & = & 5 & -x_1 & \\ x_4 & = & 8 & & -x_2 \\ x_5 & = & 10 & -x_1 & -x_2 \end{array}$$

If x_1 enters, then x_3 leaves the basis, so we pivot on

$$x_3 = 5 - x_1 \quad \text{which becomes} \quad x_1 = 5 - x_3$$

giving us the second dictionary

$$\begin{array}{ccrrr} z & = & 30 & -6x_3 & +7x_2 \\ \hline x_1 & = & 5 & -x_3 & \\ x_4 & = & 8 & & -x_2 \\ x_5 & = & 5 & +x_3 & -x_2 \end{array}$$

Now x_2 must enter the basis (since only x_2 has a positive coefficient in the z row), so x_5 leaves the basis, so we pivot on

$$x_5 = 5 + x_3 - x_2 \quad \text{which becomes} \quad x_2 = 5 + x_3 - x_5$$

giving us the third dictionary

$$\begin{array}{ccrrr} z & = & 65 & +x_3 & -7x_2 \\ \hline x_1 & = & 5 & -x_3 & \\ x_4 & = & 3 & -x_3 & +x_5 \\ x_2 & = & 5 & +x_3 & -x_5 \end{array}$$

Now x_3 must enter, and x_4 leaves the basis, so we pivot on

$$x_4 = 3 - x_3 + x_5 \quad \text{which becomes} \quad x_3 = 3 - x_4 + x_5$$

giving us the fourth dictionary

$$\begin{array}{ccrrr} z & = & 68 & -x_4 & -6x_5 \\ \hline x_1 & = & 2 & +x_4 & -x_5 \\ x_3 & = & 3 & -x_4 & +x_5 \\ x_2 & = & 8 & -x_4 & \end{array}$$

Since all the z row coefficients are negative, this gives us the optimal value $z = 68$ for $x_1 = 2$ and $x_2 = 8$ (in terms of the decision or original variables); in fact, this final dictionary is the same as the previous final dictionary.

Problem 2

(a) (There are a few ways of doing this): Since Alice takes the stochastic $\mathbf{x}$ such that $\mathbf{x}^{\mathrm{T}}\mathbf{A}$ has the possible largest minimum component over all possible stochastic $\mathbf{x}$, this minimum component is at least as big as that of any particular $\mathbf{x}$. Said otherwise: the value of "Alice announces a mixed strategy is

$$\text{AliceAnnouncesMixed}(A) = \max_{\mathbf{x} \text{ stoch}} \text{MinEntry}(\mathbf{x}^{\mathrm{T}} A);$$

hence this is at least as large as any particular value of

$$\text{MinEntry}(\mathbf{x}^{\mathrm{T}} A),$$

for any particular stochastic vector, $\mathbf{x}$; so if

$$\mathbf{x}^{\mathrm{T}} A \geq [v \ v \ \ldots \ v],$$

then

$$\mathrm{MinEntry}(\mathbf{x}^{\mathrm{T}}A) \geq v.$$

(b) We have

$$(\mathbf{x}^{\mathrm{T}}A)\mathbf{y} \geq [v\ v\ \ldots\ v]\,\mathbf{y} = vy_1 + vy_2 + \cdots + vy_n = v$$

since $\mathbf{y}$ is stochastic.

(c) Similarly any particular choice of stochastic $\mathbf{y}$ gives a $\mathrm{MaxEntry}(A\mathbf{y}) \leq w$ larger than

$$\mathrm{BettyAnnouncesMixed}(A) = \min_{\mathbf{y}\ \text{stoch}} \mathrm{MaxEntry}(A\mathbf{y}).$$

(d) (There are a few ways of doing this.) Similar to part (b) we have

$$(\mathbf{x}^{\mathrm{T}}A)\mathbf{y} = \mathbf{x}^{\mathrm{T}}(A\mathbf{y}) \leq \mathbf{x}^{\mathrm{T}} \begin{bmatrix} w \\ \vdots \\ w \end{bmatrix} = w$$

Since $\mathbf{x}$ is stochastic. Combining this with part (b) we have

$$v \leq \mathbf{x}^{\mathrm{T}}A\mathbf{y} \leq w.$$

(e) The value of the mixed strategy games is at least 1.1 (since this is the minimum of 1.2, 1.4, and 1.1) and at most 1.3 (the maximum of 1.1 and 1.3).

(f) This is not possible, for in this case the value of the game would be at least 1.1 but at most 0.9 which is impossible.

Problem 3

As discussed before, the value of the mixed strategy game is positive, so we can write this value as the maximum of v (with decision variables v and x_1) subject to $x_1 \leq 1$ and

$$[x_1\ 1-x_1] \begin{bmatrix} 1 & 3 \\ 8 & 1 \end{bmatrix} \geq [v\ v]$$

and $x_1, v \geq 0$. This gives us an initial dictionary:

$$\begin{array}{rcrrr} z & = & & & v \\ \hline x_2 & = & 1 & -x_1 & \\ x_3 & = & 8 & -7x_1 & -v \\ x_4 & = & 1 & +2x_1 & -v \end{array}$$

So v enters the basis and x_5 leaves, so we pivot on

$$x_4 = 1 + 2x_1 - v \quad \text{which becomes} \quad v = 1 + 2x_1 - x_4$$

giving the second dictionary

$$\begin{array}{rcrrr} z & = & 1 & +2x_1 & -x_5 \\ \hline x_2 & = & 1 & -x_1 & \\ x_3 & = & 7 & -9x_1 & +x_4 \\ v & = & 1 & +2x_1 & -x_4 \end{array}$$

So x_1 enters and x_3 leaves, so we pivot on

$$x_3 = 7 - 9x_1 + x_4 \quad \text{which becomes} \quad x_1 = (7/9) - (1/9)x_3 + (1/9)x_4$$

giving the third dictionary

$$\begin{array}{rcrrr}
z & = & 11/9 & -(2/9)x_3 & -(7/9)x_4 \\
\hline
x_2 & = & 2/9 & +(1/9)x_3 & -(1/9)x_4 \\
x_1 & = & 7/9 & -(1/9)x_3 & +(1/9)x_4 \\
v & = & 23/9 & -(2/9)x_3 & -(7/9)x_4
\end{array}$$

Hence $v = 23/9$ and Alice's optimal mixed strategy is $x_1 = 7/9$, $x_2 = 2/9$. It is interesting to note that Betty's optimal mixed strategy is $2/9$ and $7/9$, which are the z (or v) row coefficients in this final dictionary

Department of Computer Science, University of British Columbia, Vancouver, BC V6T 1Z4, CANADA, and Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, CANADA.

E-mail address: jf@cs.ubc.ca or jf@math.ubc.ca

URL: http://www.math.ubc.ca/~jf