

Math 340, Nov. 4

Ch. 5: Complementary Slackness

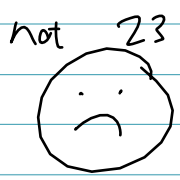
(an example of LP without LP)

Hopefully optimal bound:

$$x_1^* = 2, x_2^* = 3, z = 4x_1^* + 5x_2^* = 23$$

$$\begin{aligned} (x_1 + 2x_2 \leq 8) &\cdot 2 \\ (x_1 + x_2 \leq 5) &\cdot 0 \\ (2x_1 + x_2 \leq 8) &\cdot 1 \end{aligned}$$

$$4x_1 + 5x_2 \leq 24$$



why not 23

$$2x_1 + x_2 \leq 8 \text{ at } x_1^* = 2, x_2^* = 3$$

$$\underline{7} \leq 8 \text{ (sad face)}$$

$$\begin{aligned} \max \quad & 4x_1 + 5x_2 \text{ s.t.} \\ & x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 5 \\ & 2x_1 + x_2 \leq 8 \\ & (x_1, x_2 \geq 0) \end{aligned}$$

Turns out $x_1^* = 2, x_2^* = 3$ optimal

Vagueness!

$$z = 23 + (1 \cdot x_3 - 3 \cdot x_4)$$

$x_1 = 2$
 $x_2 = 3$
 $x_5 = 1$

$$x_5 = 8 - 2x_1 - x_2 \geq 0$$

$$x_5 = 8 - 7 > 0$$

Dual optimal

$$\begin{aligned} (x_1 + 2x_2 \leq 8) \cdot 1 &= y_1^* \\ (x_1 + x_2 \leq 5) \cdot 0 &= y_2^* \\ (2x_1 + x_2 \leq 8) \cdot 1 &= y_3^* \end{aligned}$$

At $x_1^* = 2, x_2^* = 3$

$$\begin{aligned} x_1 + 2x_2 &\leq 8 \\ \text{reads } 2 + 2 \cdot 3 &\leq 8 \text{ (equality)} \\ x_1 + x_2 &\leq 5 \\ 2 + 3 &\leq 5 \text{ (equality)} \\ 2x_1 + x_2 &\leq 8 \\ 2 \cdot 2 + 3 &\leq 8 \\ \underline{7} &\leq 8 \text{ ("slack")} \end{aligned}$$

$$y_4: 4 \leq y_1 + y_2 + 2y_3$$

$$y_5: 5 \leq 2y_1 + y_2 + y_3$$

$$y_1^* = 1, y_2^* = 3, y_3^* = 0, y_4^* = y_5^* = 0$$

Complementary Slackness

① $x_{dec} \begin{cases} x_1 \leftrightarrow y_4 \\ x_2 \leftrightarrow y_5 \end{cases} \} y_{slack}$

$x_3 \leftrightarrow y_1$

$x_4 \leftrightarrow y_2$

$x_5 \leftrightarrow y_3$

Complementary Slackness

$$\max (4x_1 + 5x_2 \text{ s.t.}) \left. \begin{array}{l} x_3 \quad (x_1 + 2x_2 \leq 8) \\ x_4 \quad (x_1 + x_2 \leq 5) \\ x_5 \quad (2x_1 + x_2 \leq 8) \end{array} \right\} \begin{array}{l} y_1 \\ y_2 \\ y_3 \end{array} \text{ Dual}$$

Optimal Solutions

decision $\begin{cases} x_1^* = 2 \text{ --- } y_4^* = 0 \\ x_2^* = 3 \text{ --- } y_5^* = 0 \end{cases}$

slack $\begin{cases} x_3^* = 0 \text{ --- } y_1^* = 1 \\ x_4^* = 0 \text{ --- } y_2^* = 3 \\ x_5^* = 0 \Rightarrow y_3^* = 0 \end{cases}$

max $10x_1$ s.t.

$x_2 \rightarrow x_1 \leq 6$

$x_3 \rightarrow x_1 \leq 6$ (toy example)

$x_4 \rightarrow x_1 \leq 6$

$x_5 \rightarrow 100x_1 \leq 600$

In opt: $x_1^* = 6, obj = 10x_1 = 60$

$x_1^* = 6$

$x_2^* = 0$

$x_3^* = 0$

$x_4^* = 0$

② If say, $x_5 \neq 0$ in optimality,

$x_5^* = 1 \Rightarrow y_3^* = 0$

i.e. we can't have both

x_5^* and $y_3^* > 0$ in optimality

can't have

(x_4^*) and (its corresp, both y variable > 0)

in optimality

... They can both be zero ...

Orig: Optimal: $x_1^* = 6$.

Dual:

$(x_1 \leq 6) \quad 16$	$(\quad) \quad 3$
$(x_1 \leq 6) \quad 0$	$(\quad) \quad 9$
$(x_1 \leq 6) \quad 0$	$(\quad) \quad 0$
$(100x_1 \leq 600) \quad 7$	$(\quad) \quad \frac{7}{100}$
$10x_1 \leq 60$	$10x_1 \leq 60$

☺

Dual: $\max 10x_1$

$x_2 \rightarrow (x_1 \leq 6) \quad y_1$

$x_3 \rightarrow (x_1 \leq 6) \quad y_2$

$x_4 \rightarrow (x_1 \leq 6) \quad y_3$

$x_5 \rightarrow (100x_1 \leq 600) \quad y_4$

$(y_1 + y_2 + y_3 + 100y_4) x_1 \leq 6y_1 + 6y_2 + 6y_3 + 600y_4$

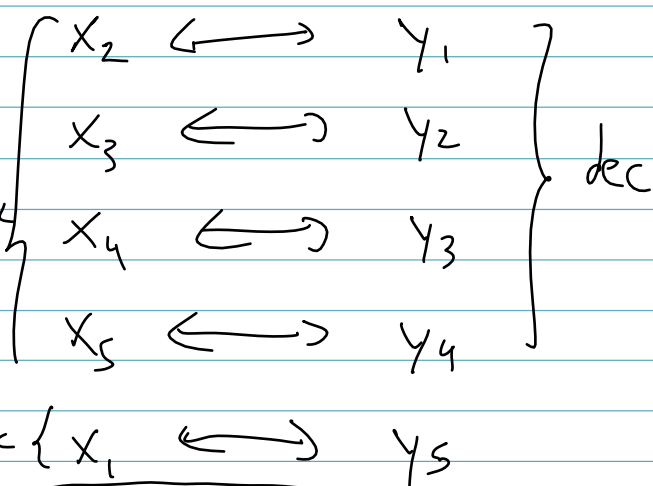
$\min 6y_1 + 6y_2 + 6y_3 + 600y_4$

s.t $10 \leq y_1 + y_2 + y_3 + 100y_4$

$x_1^* = 6$	$y_5^* = 0$	6
$x_2^* = 0$	$y_1^* = 1$	0
$x_3^* = 0$	$y_2^* = 2$	0
$x_4^* = 0$	$y_3^* = 3$	0
$x_5^* = 0$	$y_4^* = \frac{4}{100}$	0

Dual of Dual

Comp Slackness



Fact
In optimality, if $x_i \leftrightarrow y_j$
then x_i, y_j can't both
be positive

e.g.

$x_1^* = 6$	$\xrightarrow{\text{zero}}$	$y_5^* = 0$
$x_2^* = 0$	$\xleftarrow{\text{zero}}$	$y_1^* = 10$
$x_3^* = 0$	\leftrightarrow	$y_2^* = 0$
$x_4^* = 0$	\leftrightarrow	$y_3^* = 0$
$x_5^* = 0$	\leftrightarrow	$y_4^* = 0$

??

$$x_1^* = 3 \quad \Leftrightarrow \quad -y_5^* = 0$$

$$x_2^* = 2 \quad \Leftrightarrow \quad -y_5^* = 0$$

$$x_3^* = 1 \quad \Leftrightarrow \quad 0 = y_1^*$$

$$x_4^* = 0 \quad \Leftrightarrow \quad y_2$$

$$x_5^* = 0 \quad \Leftrightarrow \quad y_3$$

Say: some LP

max $4x_1 + 5x_2$

$$x_3: \quad x_1 + 2x_2 \leq 8$$

$$x_4: \quad x_1 + x_2 \leq 5$$

$$x_5: \quad 2x_1 + x_2 \leq 8$$

Guess $x_1 = 3, x_2 = 2$ at optimality

$$z = 4 \cdot 3 + 5 \cdot 2 = 22$$

If worked...

must be $0 =$

$$y_4^* = -4 + y_1^* + y_2^* + 2y_3^* \geq 0$$

$$0 = y_5^* = -5 + 2y_1^* + y_2^* + y_3^* \geq 0$$

Check...

$$\left. \begin{aligned} y_2^* + 2y_3^* &= 4 \\ y_2^* + y_3^* &= 5 \end{aligned} \right\} \quad y_3^* = -1$$
