Math 340, Nov. 4

Ch. 5: Complementary Slackness

(an example of LP without LP)

max $4x_1 + 5x_2$ s.t.

$x_1 + 2x_2 \le 8$

$x_1 + x_2 \le 5$

$2x_1 + x_2 \le 8$

$(x_1, x_2 \ge 0)$

Turns out $x_1^* = 2, x_2^* = 3$ optimal

Hopefully optimal bound:

$x_1^* = 2,\ x_2^* = 3,\ z = 4x_1^* + 5x_2^* = 23$

$(x_1 + 2x_2 \le 8) \cdot 2$

$(x_1 + x_2 \le 5) \cdot 0$

$(2x_1 + x_2 \le 8) \cdot 1$

$4x_1 + 5x_2 \le 24$

not 23

why not 23

$\underbrace{2x_1 + x_2}_{7} \le 8$ at $x_1^* = 2, x_2^* = 3$

$7 \le 8$

At $x_1^* = 2,\ x_2^* = 3$

$x_1 + 2x_2 \le 8$

reads $2 + 2\cdot 3 \le 8$ (equality)

$x_1 + x_2 \le 5$

$2 + 3 \le 5$ (equality)

$2x_1 + x_2 \le 8$

$\underbrace{2\cdot 2 + 3}_{7} \le 8$ ("slack")

$7 \le 8$

Vaguely!

$z = 23 - 1\cdot x_3 - 3\cdot x_4$

$x_1 = 2$

$x_2 = 3$

$x_5 = 1$

$x_5 = 8 - 2x_1 - x_2 \ge 0$

$x_5 = 8 - 7 > 0$

Dual optimal

$(x_1 + 2x_2 \le 8) \cdot 1 = y_1^*$

$(x_1 + x_2 \le 5) \cdot 3 = y_2^*$

$(2x_1 + x_2 \le 8) \cdot 0 = y_3^*$

y_4: $4 \le y_1 + y_2 + 2y_3$

y_5: $5 \le 2y_1 + y_2 + y_3$

$y_1^* = 1$, $y_2^* = 3$, $y_3^* = 0$, $y_4^* = y_5^* = 0$

Complementary Slackness

(1)

x_{dec} $\quad x_1 \longleftrightarrow y_4$, $\quad x_2 \longleftrightarrow y_5$ $\Big\}$ y_{slack}

$x_3 \longleftrightarrow y_1$

$x_4 \longleftrightarrow y_2$

$x_5 \longleftrightarrow y_3$

Complementary Slackness

$\max\ 4x_1 + 5x_2$ s.t. $\quad$ Dual

x_3	$(x_1 + 2x_2 \le 8)$	$\cdot y_1$
x_4	$(x_1 + x_2 \le 5)$	$\cdot y_2$
x_5	$(2x_1 + x_2 \le 8)$	y_3

Optimal Solutions

decision $\begin{cases} x_1^* = 2 & — \quad y_4^* = 0 \\ x_2^* = 3 & — \quad y_5^* = 0 \end{cases}$

slack $\begin{cases} x_3^* = 0 & — \quad y_1^* = 1 \\ x_4^* = 0 & — \quad y_2^* \ 3 \\ x_5^* = 1 & \Rightarrow \quad y_3^* = 0 \end{cases}$

$\max\ 10x_1$ s.t.

$x_2 \rightarrow \quad x_1 \le 6$

$x_3 \rightarrow \quad x_1 \le 6 \qquad$ (toy example)

$x_4 \rightarrow \quad x_1 \le 6$

$x_5 \rightarrow \quad 100x_1 \le 600$

In opt: $\quad x_1^* = 6$, $obj = 10x_1 = 60$

$x_1^* = 6$

$x_2^* = 0$

$x_3^* = 0$

$x_4^* = 0$

(2) If say, $x_5 \ne 0$ in optimality,

$x_5^* = 1 \Rightarrow y_3^* = 0$

i.e. we can't have both x_5^* and $y_3^* > 0$ in optimality

can't have (x_4^*) and (its corresp. y variable) both > 0 in optimality

..They can both be zero ...

Orig: Optimal : $x_1^* = 6$.

Dual:

$(x_1 \le 6)$ 10	( ) 3
$(x_1 \le 6)$ 0	( ) 0
$(x_1 \le 6)$ 0	( ) 0
$(100x_1 \le 600)$ 0	( ) $\frac{7}{100}$
$10x_1 \le 60$	$10x_1 \le 60$

Dual: $\max 10x_1$

$x_2 \to (x_1 \le 6)\ y_1$

$x_3 \to (x_1 \le 6)\ y_2$

$x_4 \to (x_1 \le 6)\ y_3$

$x_5 \to (100x_1 \le 600)\ y_4$

$= (y_1+y_2+y_3+100y_4)x_1 \le 6y_1+6y_2+6y_3+600y_4$

$= \min 6y_1+6y_2+6y_3+600y_4$

s.t. $10 \le y_1+y_2+y_3+100y_4$

$x_1^* = 6 \Rightarrow$	$y_5^* = 0$	6
$x_2^* = 0$	$y_1^* = 1$	0
$x_3^* = 0$	$y_2^* = 2$	0
$x_4^* = 0$	$y_3^* = 3$	0
$x_5^* = 0$	y_4^* 4/100	0

Dual of Dual

Comp Slackness

slack $\begin{cases} x_2 \longleftrightarrow y_1 \\ x_3 \longleftrightarrow y_2 \\ x_4 \longleftrightarrow y_3 \\ x_5 \longleftrightarrow y_4 \end{cases}$ dec

dec $\{ x_1 \longleftrightarrow y_5$

e.g. $x_1^* = 6 \Longrightarrow y_5^* = 0$ zero

$x_2^* = 0 \Longleftarrow y_1^* = 10$ zero

$x_3^* = 0 \longleftrightarrow y_2^* = 0$

$x_4^* = 0 \longleftrightarrow y_3^* = 0$

$x_5^* = 0 \longleftrightarrow y_4^* = 0$

Fact

In optimality, if $x_i \leftrightarrow y_j$ then x_i, y_j can't both be positive

??

$\cancel{x_1^*} = 3$ $y_4^* = 0$

$x_2^* = 2 \longleftrightarrow y_5^* = 0$

$x_3^* = 1 \longleftrightarrow 0 = y_1^*$

$x_4^* = 0 \longleftrightarrow y_2$

$x_5^* = 0 \longleftrightarrow y_3$

If worked...

must be $0 = y_4^* = -4 + \cancel{y_1} + y_2^* + 2y_3^* \geq 0$

$0 = y_5^* = -5 + \cancel{2y_1} + y_2^* + y_3^* \geq 0$

$$\left.\begin{aligned} y_2^* + 2y_3^* &= 4 \\ y_2^* + y_3^* &= 5 \end{aligned}\right\} \quad y_3^* = -1$$

Say: some LP

$\max \; 4x_1 + 5x_2$

s.t.:
$$x_1 + 2x_2 \leq 8$$
$$x_1 + x_2 \leq 5$$
$$2x_1 + x_2 \leq 8$$

Guess $\boxed{x_1 = 3, \; x_2 = 2}$ at optimality

$z = 4 \cdot 3 + 5 \cdot 2 = 22$

Check...