Math 340, Nov. 2

Ch 5: Duality:

LP: $\max \vec{c}\cdot\vec{x}$ s.t.

$$A\vec{x} \le \vec{b}$$
$$\vec{x} \ge 0$$

$\rightarrow$ dual LP

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Derivation:

$\vec{y} \cdot (A\vec{x} \le \vec{b})$, $\vec{y} \ge 0$

want

$$\boxed{\vec{c}\cdot\vec{x}} \le (\vec{y}^T A)\vec{x} \le \vec{y}\cdot\vec{b}$$

$\rightsquigarrow$ $\max -\vec{b}\cdot\vec{y}$

s.t. $-A^T\vec{y} \le -\vec{c}$

$(\vec{c}, A, \vec{b})$

$\Downarrow$

$(-\vec{b}, -A^T, -\vec{c})$

Fact: If "primal" $\max \vec{c}\cdot\vec{x}$ s.t.

$$A\vec{x} \le \vec{b}$$
$$\vec{x} \ge 0$$

is feasible and bounded, then

dual is " " "

dual objective in optimality

$$\min \vec{b}\cdot\vec{y} = \max \vec{c}\cdot\vec{x}$$

of the primal objective in optimality.

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Dictionary in primal $\rightsquigarrow$ Dictionary in dual

Primal is unbounded $\rightsquigarrow$ Dual is infeasible

$$\max \quad x_1 + x_2$$
$$\text{s.t.} \quad x_1 \le 5$$
$$x_1, x_2 \ge 0$$

for $x_1 = 0$, $x_2 = t \ge 0$, objective $= t$

$t = 10^{10^{10}}$, $x_2 = 10^{10^{10}}$, $x_1 = 0$

$\vec{x} = \begin{pmatrix} 0 \\ 10^{10^{10}} \end{pmatrix}$ is feasible $z = 10^{10^{10}}$

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we can't bound from above

$x_1, x_2 \ge 0$

$$(x_1 \le 5) \; y_1$$

$$y_1 x_1 \le 5y_1$$

$z = x_1 + x_2$ is never $\le y_1 x_1$

for any y_1 ...

We want $\min 5y_1$ s.t.

$$1 \le y_1$$
$$1 \le 0 \cdot y_1 \leftarrow \text{impossible}$$
$$y_1 \ge 0$$

max $x_1 + 10x_2$ s.t.

$x_1 + x_2 \leq 7$

$x_1, x_2 \geq 0$

Looks like $x_1 = 0$, $x_2 = 7$ ...

$z = x_1 + 10x_2$ bound via $x_1 + x_2 \leq 7$?

$x_1 + x_2 \leq 7$

$z = x_1 + 10x_2 \leq 10x_1 + 10x_2 \leq 70$

best upper bound

since $x_1 = 0$ we don't mind

Last class

max $4x_1 + 5x_2$ s.t.

$x_3 \to \quad x_1 + 2x_2 \leq 8$

$x_4 \to \quad x_1 + x_2 \leq 5$

$x_5 \to \quad 2x_1 + x_2 \leq 8$

Final dictionary:

$z = 23 - x_3 - 3x_4$

$(x_1 + 2x_2 \leq 8) \cdot 1$

$(x_1 + x_2 \leq 5) \cdot 3$

$(2x_1 + x_2 \leq 8) \cdot 0$

optimal dual solution

$z = 4x_1 + 5x_2 \leq 23$

max $4x_1 + 5x_2$ — x_1, x_2 decision

s.t. blah blah blah

x_3, x_4, x_5 slack

$4x_1 + 5x_2 = z = 23 - x_3 - 3x_4$

all slack

$(x_1 + 2x_2 \leq 8) \cdot 1$

$(x_1 + x_2 \leq 5) \cdot 3$

$z = 4x_1 + 5x_2 \leq 23$

x_1, x_2 decision

$\boxed{z = 70 - 9x_1 - 10x_3}$

$(z + 9x_1 = 70 - 10x_3)$

decision — slack

$z = x_1 + 10x_2$

$x_3 = 7 - x_1 - x_2$

x_1 enters, or x_2...

$x_2 = 7 - x_1 - x_3$

$z = x_1 + 10(7 - x_1 - x_3)$

$= 70 - 9x_1 - 10x_3$

$z = x_1 + 10x_2 = 70 - 10(7 - x_1 - x_2)$

i.e. $\qquad - 9x_1$

$x_1 + x_2 \leq 7$

$z + 9x_1 = 10x_1 + 10x_2 \leq 70$

z + non-neg comb of decision

= non-neg combo of slack

(e.g. $z = 70 - 9x_1 + 10x_3$

dec, slack

$x_3 \iff x_1 + x_2 \le 7$

$x_3 = 7 - x_1 - x_2$

So

1st ineq: some decisions $\le b_1$

2nd some other comb dec $\le b_2$

z + non-neg com of decision = $\vdots$ = optimal value

Proof that the dual LP has its min $\vec{y} \cdot \vec{b}$ = primal max $\vec{c} \cdot x$ assume ~~both are~~ primal is feasible and bounded...

Since primal is feasible and bounded, we arrive at a final dictionary:

decision, some will be basic

$$z = C - a_1x_1 - a_2x_2 - \dots - a_nx_n - a_{n+1}x_{n+1} - \dots - a_{n+m}x_{n+m}$$

slack variables

$z + a_1x_1 + \dots + a_nx_n = -a_{n+1}x_{n+1} - \dots - a_{n+m}x_{n+m}$

z + non-neg combo of $x_1, \dots, x_n$ =

2 for 1: (2) ⇒ (3)

Simplex method ⤳ dual simplex method

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Complementary slackness:

Max $4x_1 + 5x_2$

$x^* = \begin{cases} x_1 = 2 \\ x_2 = 3 \end{cases}$

1. $x_1 + 2x_2 \le 8$ — $8 \le 8$

3. $x_1 + x_2 \le 5$ — $5 \le 5$

0. $2x_1 + x_2 \le 8$ — $7 \le 8$, $7 < 8$

tr n be

If

(1) an LP is feasible & bounded ⇒ dual LP is " " "

(2) an LP is feasible & unbounded ⇒ dual LP is infeasible

(3) If dual is feasible & unbounded ⇒ original LP is infeasible

(2) ⇒ if dual LP is blah blah blah then $\underbrace{\text{dual(dual LP)}}_{\text{LP}}$ is blahhhhhh