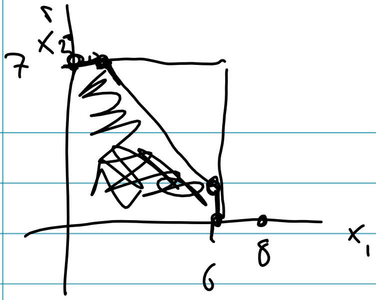
Math 340
Oct 28, 2015

Ch. 5 – Duality Theory

- Explain a lot about Alice vs. Betty
- Gives "2 for 1" deal on techniques
- Very special to linear programming

Idea: max $4x_1 + 5x_2$ s.t.

$$\begin{aligned} x_1 \quad &\le 6 \\ x_2 &\le 7 \\ x_1 + x_2 &\le 8 \\ x_1, x_2 &\ge 0 \end{aligned}$$




Guess: maybe $x_2 = 7$, $x_1 = 1$

Objective $z = 4x_1 + 5x_2$

$= 4\cdot 1 + 5\cdot 7 = 39$

Remark:

$4x_1 + 5x_2$

$\le 4\cdot 6 + 5\cdot 7$

$= 24 + 35 = 59$

$4x_1 + 5x_2 \le 59$

$(x_1 \le 6)\cdot 2$

$(x_2 \le 7)\cdot 3$

$(x_1 + x_2 \le 8)\cdot 2$

$2x_1 + 3x_2 + 2(x_1 + x_2) \le 12 + 21 + 16$

$4x_1 + 5x_2 \le 49$

$(x_1 \le 6)\cdot 0$

$(x_2 \le 7)\cdot 1$

$(x_1 + x_2 \le 8)\cdot 4$

$4x_1 + 5x_2 \le 0 + 7 + 32 = 39$

$4x_1 + 5x_2$

$= 4\underbrace{(x_1 + x_2)}_{\le 8} + \underbrace{x_2}_{\le 7}$

$\le 4\cdot 8 + 7 = 39$ !!

$(x_1 \le 6)\cdot 4$

$(x_2 \le 7)\cdot 5$

$(x_1 + x_2 \le 8)\cdot 0$

$$\left\{\begin{aligned} 4x_1 &\le 24 \\ 5x_2 &\le 35 \\ 0 &\le 0 \end{aligned}\right\} \text{ add}$$

Given:

max $4x_1 + 5x_2$ s.t.

$$x_1 \le 6$$
$$x_2 \le 7$$
$$x_1 + x_2 \le 8$$
$$x_1, x_2 \ge 0$$

We can upper bounds via:

minimize $6y_1 + 7y_2 + 8y_3$

s.t.

$$y_1 + y_3 \ge 4$$
$$y_2 + y_3 \ge 5$$
$$y_1, y_2, y_3 \ge 0$$

$$(x_1 \le 6) \cdot y_1$$
$$(x_2 \le 7) \cdot y_2$$
$$(x_1 + x_2 \le 8) \cdot y_3$$

$$(y_1 + y_3)x_1 + (y_2 + y_3)x_2 \le 6y_1 + 7y_2 + 8y_3$$

Need $y_1, y_2, y_3 \ge 0$

Need $\boxed{y_1 + y_3 \ge 4}$ $\boxed{y_2 + y_3 \ge 5}$ if

will bound $4x_1 + 5x_2$ from above

minimize $6y_1 + 7y_2 + 8y_3$

$$\vec{y}^T A\vec{x} \le \vec{y}^T \vec{b}$$

$$y_1 (x_1 \le 6)$$
$$y_2 (x_2 \le 7)$$
$$y_3 (x_1 + x_2 \le 8)$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

$6y_1 + 7y_2 + 8y_3$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

max $-6y_1 - 7y_2 - 8y_3$

s.t.

$$-y_1 - y_3 \le -4$$
$$-y_2 - y_3 \le -5$$
$$y_1, y_2, y_3 \ge 0$$

Matrix notation:

max $\vec{c} \cdot \vec{x}$ s.t.

$$A\vec{x} \le b$$
$$\vec{x} \ge 0$$

~~So from~~ In other words

minimize $\vec{y}^T \vec{b}$

s.t. $\vec{y}^T A \geq \vec{c}^T$

$\vec{y} \geq 0$

max $-\vec{b} \cdot \vec{y}$

s.t. $-\vec{y}^T A \leq -\vec{c}^T$

$(-\vec{y}^T A)^T \leq -(\vec{c}^T)^T$

$-A^T \vec{y} \leq -\vec{c}$

$\underbrace{\vec{y}^T A \vec{x}} \leq \vec{y}^T \vec{b}$

if this is $\geq \vec{c}^T \vec{x}$

then upper bound on

max $\vec{c} \cdot \vec{x}$

s.t. $A\vec{x} \leq \vec{b}$

$\vec{x} \geq 0$

$\rightarrow$ minimize $\vec{y}^T \vec{b}$ s.t.

$\vec{y}^T A \vec{x} \geq \vec{c}^T \vec{x}$

i.e. $\vec{y}^T A \geq \vec{c}^T$, $\vec{y} \geq 0$

So from:

max $\vec{c} \cdot \vec{x}$ s.t.

"primal" $A\vec{x} \leq \vec{b}$

$\vec{x} \geq 0$

max $(-\vec{b}) \cdot \vec{y}$

"dual" $(-A^T)\vec{y} \leq -\vec{c}$

$\vec{y} \geq 0$

$(\vec{c}, A, \vec{b}) \rightsquigarrow (-\vec{b}, -A^T, -\vec{c})$

again $\rightsquigarrow$

Recall $(AB)^T = B^T A^T$

similar to $(AB)^{-1} = B^{-1} A^{-1}$

((socks on) then (shoes on)$)^{-1}$

(shoes off) then (socks off)

=

max $(-\vec{b}) \cdot \vec{y}$

s.t. $(-A^T) \cdot \vec{y} \leq -\vec{c}$

$\vec{y} \geq 0$

$(\vec{c}, A, \vec{b}) \rightsquigarrow (-\vec{b}, -A^T, -\vec{c})$

$A$ vs. $-A^T$

If matrix game $M$

| | Betty | | | Alice |
|---|---|---|---|---|
| Alice | $M$ | $\frown$ | Betty | $-M^T$ |

Payoff to Alice — Payoff to Betty

Hmmm . . .

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$(A\vec{x} \le \vec{b}) \cdot \vec{y}$ — m variables

$A$ is $m \times n$ $\left\{ \begin{array}{l} n \text{ variables } x \\ m \text{ inequalities} \end{array} \right\}$

$$\max \begin{bmatrix} 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

s.t.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 6 \\ 7 \\ 8 \end{bmatrix}$$

$x_1, x_2 \ge 0$

$$\max \begin{bmatrix} -6 \\ -7 \\ -8 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

s.t.

$$-\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \le -\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$y_1, y_2, y_3 \ge 0$

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Dual (Dual) = ?

max $-6y_1 - 7y_2 - 8y_3$

s.t.

$y_4$: $-y_1 \quad - y_3 \le -4$

$y_5$: $-y_2 - y_3 \le -5$

1st Dictionary

$$y_4 = -4 + y_1 + y_3$$
$$y_5 = -5 + y_2 + y_3$$
$$w = \underline{-6}y_1 \underline{-7}y_2 \underline{-8}y_3$$

Dual = pos
= pos

$w =$ −neg coef −neg coef

max $4x_1 + 5x_2$

s.t. $x_1 \le 6$

$x_2 \le 7$

$x_1 + x_2 \le 8$

1st Dictionary

$$x_3 = 6 - x_1$$
$$x_4 = 7 - x_2$$
$$x_5 = 8 - x_1 - x_2$$
$$z = \underline{4}x_1 + \underline{5}x_2$$

2 vars = pos
= pos

$z = 39 - \_\_ - \_\_$