Simplex method (where everything goes well...)

$$\max\ 5x_1+2x_2+9x_3$$

s.t. $x_1+x_2 \le 11$

$$z = 5x_1+2x_2+9x_3$$

$\vec{x}_{slack} =$

BFS: x_1, x_2, x_3 all zero

$z = 0$

Ch. 3: GOAL:

Theorem: The simplex method (i.e. some version of it) can solve any LP.

Cutoff Midterm Friday (Oct 9)
- Mules... - 2-Phase method
- Simplex Method (at 99%...)
- Game Theory

HW due one week; solutions released immediately thereafter 3pm-ish

Deep Idea: Perturbation Method...

ε's, like x_0 in 2-Phase Method

$\max\ \vec{c}\cdot\vec{x}$ s.t. $A\vec{x} \le \vec{b}$, $\vec{x} \ge 0$

Claim: At any stage

$$z = 21 + 3x_2 + 7x_3 - 10x_4$$

since we only bring into the basis x_2 or x_3,
in the next dictionary, the value of z

(1) can never go down

(2) will always increase unless entering variable is still 0

Maybe x_1 enter the basis:

$$z = 5x_1 + 2x_2 + 9x_3$$

positive coefficient

1st BFS: $x_1 = 0 = x_2 = x_3$

hopefully fix x_2, x_3 at 0, can make x_1 positive, sending a slack variable to zero, say x_5

$\boxed{\text{2nd dictionary}}$ $\boxed{\text{2nd BFS}}$ $z = 10 - x_5 - x_2 - x_3$

x_1 now 2

$A\vec{x}_{dec} \le \vec{b}$

$\vec{x}_{slack} = \vec{b} - A\vec{x}_{dec}$

$A\vec{x}_{dec} + I \cdot \vec{x}_{slack} = \vec{b}$

$\left[A \,\vdots\, I \right] \begin{bmatrix} \vec{x}_{dec} \\ \vec{x}_{slack} \end{bmatrix} = \vec{b}$

$\begin{bmatrix} \text{part} \\ \text{of} \end{bmatrix} \vec{x}_{basic} + \begin{bmatrix} \text{other} \\ \text{part} \end{bmatrix} \vec{x}_{non\text{-}basic} = \vec{b}$

$\vec{x}_{basic}$ = uniquely expressible in terms of non-basic

Detail #1:

There are only finitely many dictionaries:

Claim: $\underbrace{x_1, x_2, x_3}_{\text{decision}}$, $\underbrace{x_4, x_5, x_6, x_7}_{\text{slack}}$

$x_4 = - \quad x_1 \quad x_2 \quad x_3$
$x_5 =$
$x_6 =$
$x_7 =$

Each dictionary is determined by its 'basic' variables...

$$\begin{array}{c} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

We want to see:
in every pivot: the basic variables are uniquely expressed in terms of non-basic:
say x_1, x_3 basic

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

max $7x_1 + 11x_2$
s.t.
$x_1 + 2x_2 \le 4$
$3x_1 + 9x_2 \le 12$

dictionary
$x_3 = 4 - x_1 - 2x_2$
$x_4 = 12 - 3x_1 - 9x_2$

$x_1 + 2x_2 + x_3 = 4$
$3x_1 + 9x_2 + x_4 = 12$

Why is $\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$ invertible

① We'll see this in revised simplex

② n decision vars
m slack vars

m ~~linear~~ linear relations / equations

$[\;\;]$

m variables to solve in m equations with $(n+m)$

Either ① there is a unique solution
② sometimes there is no solution..!

$A_{basic} \vec{x}_{basic} = \text{rhs} - \_\_\_$ vars

$$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{pmatrix} 4 \\ 12 \end{pmatrix} - \begin{bmatrix} 2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

multiply by $\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}^{-1}$:

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}^{-1} \left(\begin{bmatrix} 4 \\ 12 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 9 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} \right)}_{x_2 \text{ part?}}$$

$x_1 = \text{const} \quad x_2 \quad x_4$

$x_3 = \text{const} \quad x_2 \quad x_4$

$\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 9 \end{bmatrix}$

1st dictionary
$$\vec{x}_{slack} = \vec{b} - A \vec{x}_{dec}$$
$$z = \vec{c} \cdot \vec{x}_{dec}$$

2nd
$$z = 5 + \_\_\_\_\_$$

3rd
$$z = 6 + \_\_\_\_\_$$

4th
$$z = 14 + \_\_\_\_\_$$

$\vdots$

for example:

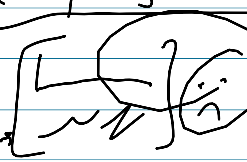
$$\cancel{x} + y = 3$$
$$x - y = 5$$

unique solution $x = 4,\ y = -1$

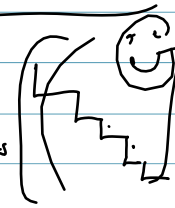
$$\left.\begin{aligned} x + y &= 3 \\ 2x + 2y &= 6 \end{aligned}\right\} \text{ infinitely many}$$

$$\left.\begin{aligned} x + y &= 3 \\ 2x + 2y &= 5 \end{aligned}\right\} \text{ 0 solutions}$$

2 eq
2 unknowns



5 eq
5 unknowns



Last problem in simplex method:

$$\max 8x_1$$

s.t.

$$\begin{cases} x_1 \le x_2 \\ x_2 \le x_3 \\ x_3 \le 12 \end{cases}$$

$$x_1, x_2, x_3 \ge 0$$

$$z = 8x_1$$

$$x_4 = 0 - x_1 + x_2$$

$$x_5 = 0 - x_2 + x_3$$

$$x_6 = 12 - x_3$$

What happens?

Want x_1 to increase

$$z = 8x_1$$

but no progress 

$$x_4 = 0 - x_1 + x_2$$

pivot

$$\boxed{x_1 = 0 - x_4 + x_2}$$

progress??

$$z = 8x_1$$

$$= 0 - 8x_4 + 8x_2$$

Trick:

$$z = 8x_1$$

$$x_4 = \varepsilon - x_1 + x_2$$

$$x_5 = \varepsilon^2 - x_2 + x_3$$

$$x_6 = 12 + \varepsilon^3 - x_3$$

(Think of ε teeny tiny > 0

$\varepsilon^3 \ll \varepsilon^2 \ll \varepsilon$)

x_1 enter:

$$x_4 = \varepsilon - x_1 + x_2 \quad !!!$$

$$x_1 = \varepsilon - x_4 + x_2 \quad !!$$

$$z = 8\varepsilon - 8x_4 + 8x_2$$

Instead z from 0 to 0

We have " " 0 to 8ε