MATH 340: FINAL REMARKS ON MATRIX GAMES AND POKER

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Contents

In this document we will analyze the BIG POKER QUESTION in Section 8 of the handout "Math 340: Matrix Games and Poker." We will also describe matrix games where each column is a convex or concave down function of the row number.

1. From 2^{52} to 53

Alice can bet or fold on any one of 52 cards. However, if she chooses to bet on some number, say b, of the 52 cards, it is clear that she does best by betting on the top b cards and folding on the bottom $52-b$ cards. So for $b = 0, 1, \ldots, 52$ (53 cases in total) it suffices to consider the payout to Alice if she bets on the top b cards and folds on the others. Let us perform this computation.

If Betty chooses the pure strategy "fold," then $b/52$ times Alice wins one dollar, and $(52-b)$ Alice looses one dollar, for a total payout of

$$f_{\text{fold}}(b) = (b/52) + (-1)(52-b)/52 = 2b/52 - 1.$$

If Betty calls, then again $(52-b)/52$ times Alice looses one dollar by folding; but in the $b/52$ times that Alice bets, Alice either wins two

Research supported in part by an NSERC grant.

dollars or looses two dollars: Alice's chance of winning will be

$$51/51,\ 50/51,\ 49/51,\ldots,(52-b)/51,$$

depending on which of the top b cards she draws; so on average Alice's chance of winning is the average of the above numbers, which is 1/2 the highest, namely 51/51, and 1/2 the lowest; call this number

$$p_b = (1/2)(51/51) + (1/2)(52-b)/51 = (103-b)/102;$$

so $b/52$ times Alice's payout is 2 with probability p_b, and -2 with probability $1-p_b$. So Alice's overall payout is

$$f_{\rm call}(b) = (b/52)\bigl(2p_1 + (-2)(1-p_b)\bigr) + (-1)(52-b)/52$$

$$= (1/52)\bigl(b(4p_b-2)-52+b\bigr) = (1/52)\bigl(4bp_b-52-b\bigr) = (1/52)\bigl(4b(103-b)/102-52-b\bigr)$$

$$= (1/52)\bigl(-4b^2/102 + b(4\cdot 103/102 - 1) - 52\bigr)$$

$$= (1/52)\bigl(-2b^2/51 + b(155/51) - 52\bigr).$$

In other words, the b-th row of this matrix game is

$$f(b) = (1/52)[\ \ -2b^2/51 + b(155/51) - 52 \qquad 2b-52\ \ ].$$

We notice that the second component is a linear function of b, and the first is a concave down function of b (i.e., a quadratic with negative b^2 coefficient).

2. Convex and Concave Down Games

At this point we will make some general remarks about convex and concave down functions and matrix games whose b-th row are convex or concave down functions.

Recall that a twice differentiable function, f, is *convex* if $f''(x)\ge 0$ for all x (or all x in some interval where f is defined). For example, the function $f(x) = ax^2+bx+c$ is convex if $f''(x) = 2a$ is non-negative. For such a function we have that for any fixed $x_0 < x < x_1$

$$f(x) \le f(x_0)\alpha + f(x_1)(1-\alpha),$$

where $\alpha = (x_1-x)/(x_1-x_0)$ (so $x_0<x<x_1$ implies that both α and $1-\alpha$ are positive). This means that if $f_1(x), f_2(x)$ are both convex functions, then

$$[f_1(x)\ f_2(x)] \le [f_1(x_0)\ f_2(x_0)]\alpha + [f_1(x_1)\ f_2(x_1)](1-\alpha).$$

It follows that if Alice is playing a matrix game whose columns are all convex functions of the row number, then any row that is not the top or bottom row is worse for her than some mixture of the top and bottom row.

The situation is reverse for concave down functions; f is *concave down* if $-f$ is convex. This means that for any fixed $x_0 < x < x_1$

$$f(x) \geq f(x_0)\alpha + f(x_1)(1-\alpha)$$

with $\alpha = \alpha(x_0, x, x_1)$ as above. If a matrix game has all columns being concave down, then if we play rows $r_1, r_2 \ldots, r_m$ (which must be integers) with frequencies $\alpha_1, \alpha_2, \ldots, \alpha_m$ respectively, then setting

$$x = r_1\alpha_1 + r_2\alpha_2 + \cdots + r_m\alpha_m$$

(i.e., the weighted average of the row numbers), then playing row x is better than the weighted average. However, if x is not an integer then there is no row x; still, if $n < x < n+1$ with n an integer, and so x is a weighted mixture of n and $n+1$, then playing this mixture of rows n and $n+1$ is better than the combination of rows $r_1, \ldots, r_n$ above.

To summarize: if the columns of a matrix game are convex functions of the row number, then Alice is always best off playing some mixture of the top and bottom row; if they are concave down functions, then she is best off playing some mixture of rows n and $n+1$ for some n, i.e., playing some two consecutive rows.

3. Analyzing the Big Poker Game

From the above considerations, it follows that for the big poker game above, Alice should play some two consecutive rows. Since the payout is

$$f(b) = (1/52)[\ -2b^2/51 + b(155/51) - 52 \quad 2b - 52\ ],$$

this means that we shoud find the b such that

$$\min(-2b^2/51 + b(155/51) - 52, 2b - 52)$$

is maximized, and play row b if b is an integer, and otherwise play rows n and $n+1$ where $n < b < n+1$.

At this point it can be helpful to plot the two components of $f(b)$ just to figure out how to argue most quickly; a plot shows that the quadratic is smaller than the linear function once b is a bit bigger than 26, and the quadratic keeps increasing for a while. Given this, we can make the precise reasoning and computation below to find Alice's optimal strategy and the value of the various games.

This minimum is no larger than

$$f_1(b) = \Big(-2b^2/51 + b(155/51) - 52\Big)(1/52)\ ,$$

whose maximum value is attained when its derivative in b is zero, i.e., when $-4b+155 = 0$, i.e., when $b = 155/4$, so $38 < b < 39$. Since $f_2(b) = 2b - 52$ is greater than $f_1(b)$ at $b = 38$ and $b = 39$, we see that Betty will play column 1 entirely at these values. Since f_1 is a quadratic, and $155/4 = 38.75$ is closer to 39, we know $f_1(39) \geq f_1(38)$. It follows that Alice will play the 39 highest cards in her optimal strategy, and her payout will be

$$(1/52)f_1(39) = 0.13235294\ldots$$

4. Learning Goals / Sample Exam Problems

If you are told that the columns of a matrix game are all convex functions of their row numbers, or all concave down functions of their row numbers, you should be able to apply Section 2 to greatly simplify the matrix game. You should be able to do the same—by duality—if this holds with "column" and "row" interchanged.

An example is given on the Final Exam for Math 340, December 2014, Problem 5(g). Here are other examples:

(1) Find the value of the matrix game

$$A = \begin{bmatrix} 1 & 4 & 9 & 16 & 25 & 36 & 49 \\ 49 & 36 & 25 & 16 & 9 & 4 & 1 \end{bmatrix}$$

(2) Find the value of the 7×3 matrix game with b-th row having entries

$$b^2, \quad 2(5-b)^2, \quad (6-b)^2.$$

[Hint: notice that these three functions are quadratic with a positive b^2 coefficient.]

5. Answers to Sample Exam Problems

The answers to the sample exam problems in this section are on the next page.

(1) The dual game is

$$A = \begin{bmatrix} -1 & -49 \\ -4 & -36 \\ -9 & -25 \\ -16 & -16 \\ -25 & -9 \\ -36 & -4 \\ -49 & -1 \end{bmatrix}$$

(now Betty plays the rows) Since the columns are concave down functions, namely $-b^2$ and $-(b-8)^2$, Betty does best to play a single value for b if this value is an integer, and otherwise to play the two integer values nearest b if b is not an integer. If Betty plays row b, then Alice (now playing the columns), will choose the column with the lesser value; the value to Betty is therefore

$$\min(-b^2, -(b-8)^2) = \min(-b^2, -b^2 + 16b - 64),$$

the value $-b^2$ is smaller for $16b - 64 \geq 0$, i.e., $b \geq 4$, and larger for $b \leq 4$. For $b \geq 4$, $-b^2$ takes on the maximum value at $b = 4$, which is the value -16, and for $b \leq 4$, $-(b-8)^2$ takes on its maximum value at $b = 4$ which is also -16. Hence Betty will play row number 4 100% of the time. It follows that in the original game, Betty will play column number 4 100% of the time.

(2) The three columns are convex functions of b; hence Alice can play only the top row (row 1) and bottom row (row 7) in an optimal strategy. Hence the matrix game reduces to the game with only rows 1 and 7, i.e.:

$$\begin{bmatrix} 1 & 32 & 25 \\ 49 & 8 & 1 \end{bmatrix}$$

Since column 3 dominates column 2 for Betty, the game reduces to

$$\begin{bmatrix} 1 & 25 \\ 49 & 1 \end{bmatrix}$$

which is a 2×2 matrix game which we know how to solve (it turns out that Alice will play $2/3$ of row 1 and $1/3$ of row 7, and the value of the game is 17).

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