The Spotlight Problem is Fixed-Parameter Tractable

Jesse Bingham and Stephane Durocher

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Given a fixed angle α, let α-SPOTLIGHT be the decision problem that asks, given an interval of the x-axis and a set of points (all above the x-axis), if spotlights that diverge at angle α can be oriented at each point such that the interval is entirely illuminated. Let

$$k(\alpha) = \begin{cases} \lceil \cot(\alpha/2) \rceil + 2\lceil \pi/(4\alpha) \rceil & \text{if } \alpha < \pi/2 \\ 2 & \text{if } \alpha \geq \pi/2 \end{cases}$$

Theorem 1 *For any α, α-SPOTLIGHT can be decided in constant time.*

Proof: Given the input set of n points and the finite interval I, a constant time algorithm works as follows. If $n \geq k(\alpha)$, return TRUE. Otherwise, try each of the $n!$ orderings of the points to illuminate I. Since $n! \leq k(\alpha)!$ and $k(\alpha)!$ is constant, this takes constant time. Correctness follows from Lemma 1 below. □

Lemma 1 *Any set of $k(\alpha)$ points can illuminate the entire x-axis.*

Proof: Let S be a set of $k(\alpha)$ points. If $\alpha \geq \pi/2$ then $k(\alpha) = 2$ and the proof is easy. Otherwise, select (x_0, y_0) (not necessarily a point in S) such that S can be partitioned into the three sets H, L, and R as depicted in Fig. 1. In particular we require that

- $|H| = \lceil \cot(\alpha/2) \rceil$
- $|L| = |R| = \lceil \pi/(4\alpha) \rceil$
- for all $(x, y) \in H$, $y \geq y_0$
- for all $(x, y) \in L$, $x \leq x_0$ and $y \leq y_0$
- for all $(x, y) \in R$, $x \geq x_0$ and $y \leq y_0$

Such a partitioning can always be done. We will show that the points of R, H, and L can respectively illuminate the intervals $(-\infty, x_0 - y_0]$, $[x_0 - y_0, x_0 + y_0]$, and $[x_0 + y_0, \infty)$.

We first prove the statement about H. Note that the interval $[x_0 - y_0, x_0 + y_0]$ has width $2y_0$. Consider any point (x, y) in H. No matter what direction the spotlight at (x, y) is oriented, it always illuminates an interval that is at least $2y\tan(\alpha/2)$ wide. Since $y \geq y_0$, we have $2y\tan(\alpha/2) \geq 2y_0\tan(\alpha/2)$. It follows that any $\lceil 2y_0/(2y_0\tan(\alpha/2)) \rceil = \lceil \cot(\alpha/2) \rceil$ such points can illuminate the entire interval $[x_0 - y_0, x_0 + y_0]$ (in fact *any* ordering of the spotlights in H across this interval is sufficient).

We now argue that the points of L can illuminate the infinite interval $[x_0 + y_0, \infty)$; the analogous statement about R follows by an analogous argument. Conceptually, we translate all spotlights of L to be collocated at (x_0, y_0). Then clearly the spotlights need only illuminate an angle of $\pi/4$ from (x_0, y_0) in order to illuminate all of $[x_0 + y_0, \infty)$. Since $|L| = \lceil \pi/(4\alpha) \rceil$, there are sufficiently many spotlights to accomplish this. Note that any ordering will do; we fix an arbitrary ordering, hence making each spotlight "responsible" for illuminating some interval (one of which will be an infinite interval).

Now consider translating a spotlight away from (x_0, y_0) to its "rightful" position (x, y). Since $y \leq y_0$ and $x \leq x_0$, this can be accomplished by first translating the spotlight downwards to the point (x_0, y) and then leftwards to (x, y). Let I be the interval the spotlight is responsible for illuminating. By Lemma 2, I can be illuminated from (x_0, y). Clearly, this implies that I can also be illuminated from (x, y).

□

Lemma 2 *Let l_1 be an α-spotlight ($0 < \alpha < \pi/2$) that initially lies a distance y_1 above the x-axis. Let o denote the point on the x-axis directly below l_1 and let a denote the near end of region illuminated by l_1. Let $\theta_1 = \angle o l_1 a$. If $\theta_1 \geq \pi/4$ and l_1 moves toward o while a remains fixed, then the length of the region illuminated by l_1 does not decrease.*

Proof: Let b_1 denote the far end of the region illuminated by l_1. Let $x_1 = ||b_1 - o||$. Let l_2 denote the new position of the spotlight between l_1 and o. Let y_2, θ_2, b_2, and x_2 denote the analogous points and lengths for l_2. Let $w = ||a - o||$. See Fig. 2.

The lemma is easy if l_1 illuminates an infinite region (i.e., $b_1 = \infty$) since then $\theta_1 + \alpha \geq \pi/2$, implying that $\theta_2 + \alpha \geq \pi/2$, since $\theta_2 \geq \theta_1$. Also, the lemma clearly holds if b_1 is finite but $b_2 = \infty$. Thus we focus our attention on the case of b_1 and b_2 both being finite, which implies both $\theta_1 + \alpha < \pi/2$ and $\theta_2 + \alpha < \pi/2$.

The length y_1 can be expressed in terms of w and θ_1.

$$y_1 = w \sin\frac{\pi}{2}\left(\cot\theta_1 + \cot\frac{\pi}{2}\right) = w\cot\theta_1 \quad \text{(AAS theorem).} \tag{1}$$

Similarly,

$$y_2 = w\cot\theta_2. \tag{2}$$

We now express x_1 in terms of w, θ_1, and α.

$$\begin{aligned} x_1 = &\quad y_1 \tan(\theta_1 + \alpha) \\ = &\quad \tfrac{w\tan(\theta_1+\alpha)}{\tan\theta_1} \qquad \text{(since } y_1 = w\cot\theta_1\text{)} \\ = &\quad \tfrac{w(\tan\theta_1+\tan\alpha)}{\tan\theta_1(1-\tan\alpha\tan\theta_1)}. \end{aligned} \tag{3}$$

Similarly,

$$x_2 = \frac{w(\tan\theta_2 + \tan\alpha)}{\tan\theta_2(1-\tan\alpha\tan\theta_2)}. \tag{4}$$

We now show $x_2 \geq x_1$. Let $f(\alpha) = x_2 - x_1$. Specifically,

$$f(\alpha) = \frac{w(\tan\theta_2 + \tan\alpha)}{\tan\theta_2(1-\tan\alpha\tan\theta_2)} - \frac{w(\tan\theta_1 + \tan\alpha)}{\tan\theta_1(1-\tan\alpha\tan\theta_1)}. \tag{5}$$

We need to show $f(\alpha) \geq 0$ for $\alpha \in (0, \pi/2)$. Observe that $f(0) = 0$. We examine the derivative of f with respect to α:

$$f'(\alpha) = \overbrace{w}^{t_1} \frac{\overbrace{\cos\alpha}^{t_2}\overbrace{[\cos\theta_1\sin\theta_1 - \cos\theta_2\sin\theta_2]}^{t_3} + \overbrace{2\sin\alpha}^{t_4}\overbrace{[\cos^2\theta_1 - \cos^2\theta_2]}^{t_5}}{\underbrace{\cos^3\alpha\cos^2\theta_1\cos^2\theta_2}_{t_6}} \tag{6}$$

Since $\alpha, \theta_1, \theta_2 \in [0, \pi/2]$, observe that terms t_2, t_4, and t_6 are non-negative. Term $t_1 = w = ||a - o|| > 0$. Term $t_5 = \cos^2\theta_1 - \cos^2\theta_2 > 0$ since $\pi/4 < \theta_1 < \theta_2 < \pi/2$ implies $\cos\theta_1 > \cos\theta_2 > 0$. As for term t_3,

$$t_3 = \cos\theta_1\sin\theta_1 - \cos\theta_2\sin\theta_2 \geq 0, \text{ for all } \pi/4 \leq \theta_1 < \theta_2 < \pi/2. \tag{7}$$

This follows from the fact that

$$\frac{\partial}{\partial x}\cos x\sin x = 2\cos^2 x - 1 \leq 0, \text{ for all } \pi/4 \leq \theta_1 < \theta_2 < \pi/2. \tag{8}$$

Therefore, $f'(\alpha) \geq 0$ for all $\alpha \in [0, \pi/2]$ and all $\theta_1, \theta_2 \in [\pi/4, \pi/2]$. Since $f(0) \geq 0$, therefore $f(\alpha) \geq 0$ for all $\alpha \in [0, \pi/2]$ and all $\theta_1, \theta_2 \in [\pi/4, \pi/2]$.
□

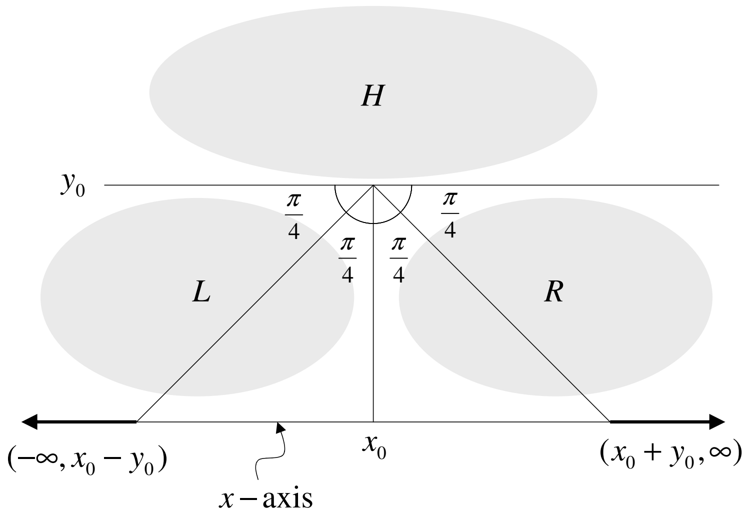

Figure 1: Partitioning of the spotlights used in the proof of Lemma 1

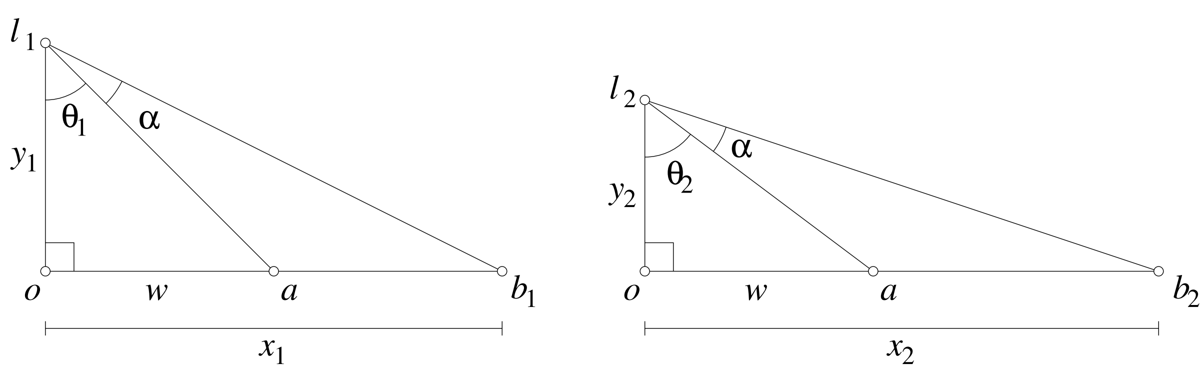

Figure 2: The various lengths, points, and angles discussed in the proof of Lemma 2