Algorithms for Pure Nash Equilibria in Weighted Congestion Games

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Outline

• Game theory background.
• Bird’s eye view of the paper.
• Weighted congestion games.
• Overview of the algorithm.
• Experimental design and empirical results.
• Comments and criticism.
Game Theory

- Models interaction between multiple agents in a structured system.

- Defined by:
  - A set of players.
  - A set of strategies for each player.
  - A payoff function for each player (a function of the strategy chosen).
Game Theory

- At each step of a game, each player is allowed to change strategies.
- Each player aims to maximise their own payoff function.
Game Theory

• A pure strategy for a given player uses only a single strategy at each step from the available set.

• A mixed strategy for a given player is a probability distribution over the set of available strategies.

• This paper only deals with pure strategies.
Game Theory

- A Nash Equilibrium is where:
  - No player can change strategies to improve their own payoff function.
  - Must assume the strategies of other players stay fixed.
Game Theory

- A Nash equilibrium is guaranteed to exist when players can use mixed strategies.
- If all players use pure strategies, a pure Nash equilibrium may exist.
Bird’s Eye View

- Weighted congestion games model the experience of users in a shared network.
- A pure Nash equilibrium always exists in these games.
- No mathematical proof that a pure Nash Equilibrium is computable in polynomial time for all instances.
Congestion Games

• Given a directed network $G = (V,E)$

• Every player wants to route traffic from a source node to a sink node in the network.

• If these source/sink nodes are the same for every player, we have a single commodity network congestion game.

• Strategy sets assumed to be equal.
Congestion Games

For each different length, finally, Network 9 is an arbitrary non layered network.

6.2.2 Distributions of Weights.

For each network, we simulated the algorithm Nashify() for n = 10, 11, 100 users. Obviously, if users’ weights are polynomial in n, then the algorithm will definitely terminate after a polynomial number of steps. Based on this fact, as well as on Proposition 4.1, we focused on instances where some users have exponential weights. More specifically, we...
Congestion Games

- Each player has a set of paths from their source node to their sink node.
  - These are the strategies.
- The payoff for a given strategy is based on the number of players sharing edges.
Weighted Version

- Each player can now demand more than one unit of traffic on a link.
- The delay on an edge is now a function of the demands of each user sharing that edge.
The Problem

- This paper considers only weighted, single-commodity network congestion games.

- Edge delays are allowed to be either polynomial or exponential in their loads (the sum of the demands).
Theoretical Results

- Proof is given that at least one pure Nash equilibrium always exists for these games.
- One of these equilibria can be computed in time polynomial in the number of players and the magnitude of the weights.
Theoretical Results

• It is conjectured by the authors that a pure Nash equilibrium is computable in polynomial time.

• Even when the edge delays are exponential.
The Algorithm

Algorithm Nashify(G, (wi)i∈N, ω)

Input: △ network G = (V, E) with a unique source–destination pair (s, t)
△ a set N = {1, . . . , n} of users, each user i having weight wi

Output: configuration ω which is a pure Nash equilibrium

1. begin
2. select an initial configuration ω = (ω1, . . . , ωn)
3. while ∃ user i that is unsatisfied
4. w_i := Shortest_Path_i(ω_{−i})
5. return ω
6. end
Experimental Design

• Nashify() was implemented in C++ using data structures in the LEDA library.

• Nine different networks of varying structure were used.

• Nashify() was run on each network, with \{10, 11, ..., 100\} players.
Experimental Design

- Two different methods for choosing an initial set of strategies.
- Four different distributions of weights.
Initial Strategies

- Random Allocation
  - Each user assigns traffic on an s-t path chosen uniformly at random.

- Shortest-Path
  - Users sorted in non-increasing order of their demands.
  - Each selects the best possible s-t path, in order.
Weight Distributions

- Four different allocations of weights were examined.
  
  1. 10% of players have weight $10^{n/10}$ and 90% of players have weight 1.
  
  2. 50% of players have weight $10^{n/10}$ and 50% of players have weight 1.
  
  3. 90% of players have weight $10^{n/10}$ and 10% of players have weight 1.
  
  4. Each player has a weight selected uniformly at random from $[1, 10^{n/10}]$. 
Networks Used

Fig. 1. Network 1.

Fig. 2. Network 2.

Fig. 3. Network 3.
Networks Used

Fig. 4. Network 4.

Fig. 5. Network 5.

Finally, Network 9 is an arbitrary nonlayered network.

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Networks Used

Fig. 7. Network 7.
Fig. 8. Network 8.
Fig. 9. Network 9.
Results

• Evidence suggests polynomial scaling on these nine networks.

• The shortest path allocation appears to dominate the random allocation.

• The authors conjecture that Nashify() will find a pure Nash equilibrium in a polynomial number of steps for any instance.
Results

• For weight distributions 1-3, \#steps/n bounded above by \log(W).

• Implies \O(n\log(W)) runtime.

• For weight distribution 4, \#steps/n bounded above by n\log(W).

• Implies \O(n^2\log(W)) runtime.
Fig. 13. Experimental results for Network 4.
Some Criticism

• May want to repeat outside of the linear spread of players (10-100).

• Perhaps try n=200 and n=500 just to confirm.

• The networks tested had a narrow spread in terms of number of nodes.

• What happens if we double the number of nodes in the same structures?
Some Criticism

• The experimental environment is never described in any detail whatsoever.

• The computation time is measured in terms of steps, with each step assumed to be a single greedy path selection.

• Should at least mention the basic machine characteristics for reproducibility.
Some Criticism

• The log(W) comparison for each network was different.

• Compared against log(W), nlog(W), 2log(W), (n/3)log(W).

• Made comparing between networks difficult.
Some Criticism

• This appears to be a manual guess of the fit for each network structure.

• Would have been more informative to do an automatic fit and compare between the structures.
Questions?