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Preconditioners for the discretized time-harmonic Maxwell equations in mixed form

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SUMMARY

We introduce a new preconditioning technique for iteratively solving linear systems arising from finite element discretization of the mixed formulation of the time-harmonic Maxwell equations. The preconditioners are motivated by spectral equivalence properties of the discrete operators, but are augmentation free and Schur complement free. We provide a complete spectral analysis, and show that the eigenvalues of the preconditioned saddle point matrix are strongly clustered. The analytical observations are accompanied by numerical results that demonstrate the scalability of the proposed approach. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

We introduce new preconditioners for linear systems arising from finite element discretization of the mixed formulation of the time-harmonic Maxwell equations in lossless media with perfectly conducting boundaries [1–4]. The following model problem with constant coefficients is

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considered: find the vector field u and the multiplier p such that

$$\begin{aligned} \nabla \times \nabla \times u - k^2 u + \nabla p &= f \quad \text{in } \Omega \\ \nabla \cdot u &= 0 \quad \text{in } \Omega \\ u \times n &= 0 \quad \text{on } \partial\Omega \\ p &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1}$$

Here, $\Omega \subset \mathbb{R}^3$ is a simply connected polyhedron domain with a connected boundary $\partial\Omega$ and n denotes the outward unit normal on $\partial\Omega$. The datum f is a given generic source (not necessarily divergence free), and the wave number satisfies $k^2 = \omega^2 \varepsilon\mu$, where $\omega \geqslant 0$ is the frequency, and ε and μ are positive permittivity and permeability parameters. We assume that k^2 is not a Maxwell eigenvalue and that

$$k^2 \ll 1$$

The introduction of the scalar variable p guarantees the stability and well-posedness of the equations as k tends to 0, including the limit case $k = 0$; see the discussion in [2, Section 3].

Finite element discretization using Nédélec elements of the first kind [5] for the approximation of the vector field and standard nodal elements for the multiplier yields a saddle point linear system of the form

$$\begin{pmatrix} A - k^2 M & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix} \tag{2}$$

where now $u \in \mathbb{R}^n$ and $p \in \mathbb{R}^m$ are finite arrays representing the finite element approximations, and $g \in \mathbb{R}^n$ is the load vector associated with f. The matrix $A \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite with nullity m, and corresponds to the discrete curl–curl operator; $B \in \mathbb{R}^{m \times n}$ is a discrete divergence operator with full row rank, and $M \in \mathbb{R}^{n \times n}$ is the vector mass matrix.

It is possible to decouple (2) into two separate problems, using the discrete Helmholtz decomposition [6, Section 7.2.1]. For p we obtain a standard Poisson equation, for which many efficient solution methods exist. Then, once p is available, the high nullity of the discrete curl–curl operator in the resulting equation for u can be dealt with by applying a procedure of augmentation: the matrix A is replaced by $A_W = A + B^{\mathrm{T}} W^{-1} B$, where $W \in \mathbb{R}^{m \times m}$ is a weight matrix, chosen so that A_W is symmetric positive definite. This does not change the solution, due to the divergence-free condition $Bu = 0$. Popular choices for W that have been considered in the literature are scaled identity matrices or lumped mass matrices; see [7, pp. 319–320] and references therein. A similar approach in the context of finite volume methods has been proposed in [8].

We note that if $k \neq 0$, a direct approach based on solving $(A - k^2 M)u = g$ automatically enforces $Bu = 0$, provided the right-hand side is divergence free. A multigrid technique for this case has been proposed in [9]. A regularization technique is introduced in [10] to deal with the case $k = 0$, whereby A is replaced by $A + \sigma M$, where σ is a regularization parameter. Algebraic multigrid is shown to converge even for small σ. The solution is divergence free for divergence-free data, but it changes with the parameter.

Leaving the saddle point system intact is a viable approach that works naturally for the limiting case $k = 0$, which is our main interest in this paper. The saddle point matrix does not have to be modified or regularized even if its $(1, 1)$ block is singular, and its structure lends itself to effective block preconditioners. Indeed, there are several robust solution methods available for solving saddle point systems [11].

An Uzawa-type algorithm for the saddle point system, coupled with a domain decomposition approach, has been proposed in [12]. The original system is transformed into a new system by augmentation with the scalar Laplacian as a weight matrix, and it is shown that the condition number of the resulting preconditioned system grows logarithmically with respect to the ratio between the subdomain diameter and the mesh size. The method incorporates augmentation and is parameter dependent. Its convergence properties rely on extreme eigenvalues of the augmented Schur complement, which may be difficult to evaluate.

In this paper we introduce a new block diagonal preconditioning technique for the iterative solution of the saddle point linear system. While it is motivated by spectral equivalence properties similar to those in [12] and by augmentation considerations, the actual preconditioners are augmentation free and parameter free. Furthermore, convergence of iterative solvers does not depend on a Schur complement. We show several equivalence properties of the matrices, and present spectral bounds based on the stability constants of the differential operators.

Each iteration of our scheme requires solving for $A + \gamma M$, where $\gamma > 0$ is given. For such systems solution techniques with linear complexity are available; see [10, 13–15] and references therein. We show that the spectral distribution of the preconditioned matrices is favourable for Krylov subspace solvers in terms of clustering of eigenvalues. We also derive explicit expressions for the eigenvectors in terms of the null vectors of the discrete operators A and B, making the convergence analysis complete.

Our numerical results indicate that the proposed technique scales extremely well with the mesh size, both on uniformly and locally refined meshes. In this paper we only focus on the performance of the outer solver, and do not consider computational issues related to how to solve the inner iterations associated with (implicit) inversion of the preconditioner.

The remainder of the paper is structured as follows. In Section 2 we present the mixed finite element formulation, make some necessary definitions, and discuss the algebraic properties of the discrete operators. In Sections 3 and 4 we discuss spectral equivalence and augmentation. In Section 5 we introduce and analyse the proposed preconditioning technique. In Section 6 we provide numerical examples that confirm the analysis and demonstrate the scalability of our approach. Finally, in Section 7 we draw some conclusions.

2. MIXED FINITE ELEMENT FORMULATION

In this section we provide details on the finite element formulation leading to the saddle point system (2).

2.1. Discretization

To discretize (1), we consider conforming and shape-regular partitions $\mathcal{T}_h$ of Ω into tetrahedra $\{K\}$. We denote the diameter of the tetrahedron K by h_K for all $K \in \mathcal{T}_h$ and define $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{P}_\ell(K)$ be the space of polynomials of degree ℓ on K and let $\mathcal{N}_\ell(K)$ be the space of Nédélec vector polynomials of the first kind [5, 6]. The index ℓ is chosen so that $\mathcal{P}_{\ell-1}(K)^3 \subset \mathcal{N}_\ell(K) \subset \mathcal{P}_\ell(K)^3$. For $\ell \geqslant 1$, the finite element spaces for the approximation of the electric field and the

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multiplier are taken as

$$V_h = \{v_h \in H_0(\text{curl}) \mid v_{h|K} \in \mathcal{N}_\ell(K), K \in \mathcal{T}_h\}$$

$$Q_h = \{q_h \in H_0^1(\Omega) \mid q_{h|K} \in \mathcal{P}_\ell(K), K \in \mathcal{T}_h\}$$

Here, we use the Sobolev space

$$H_0(\text{curl}) = \{ v \in L^2(\Omega)^3 : \nabla \times v \in L^2(\Omega)^3 ,\ v \times n = 0 \text{ on } \partial\Omega \}$$

We consider the following finite element formulation: find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} \int_\Omega (\nabla \times u_h) \cdot (\nabla \times v_h)\,\mathrm{d}x - k^2 \int_\Omega u_h \cdot v_h\,\mathrm{d}x + \int_\Omega v_h \cdot \nabla p_h\,\mathrm{d}x &= \int_\Omega f \cdot v_h\,\mathrm{d}x \\ \int_\Omega u_h \cdot \nabla q_h\,\mathrm{d}x &= 0 \end{aligned} \tag{3}$$

for all $(v_h, q_h) \in V_h \times Q_h$.

To transform (3) into matrix form, let $\langle \psi_j \rangle_{j=1}^n$ and $\langle \phi_i \rangle_{i=1}^m$ be standard finite element bases for the spaces V_h and Q_h, respectively:

$$V_h = \text{span}\langle \psi_j \rangle_{j=1}^n, \quad Q_h = \text{span}\langle \phi_i \rangle_{i=1}^m \tag{4}$$

Define

$$A_{i,j} = \int_\Omega (\nabla \times \psi_j) \cdot (\nabla \times \psi_i)\,\mathrm{d}x, \quad 1 \leqslant i, j \leqslant n$$

$$M_{i,j} = \int_\Omega \psi_j \cdot \psi_i\,\mathrm{d}x, \quad 1 \leqslant i, j \leqslant n$$

$$B_{i,j} = \int_\Omega \psi_j \cdot \nabla \phi_i\,\mathrm{d}x, \quad 1 \leqslant i \leqslant m, \ 1 \leqslant j \leqslant n$$

and let $A \in \mathbb{R}^{n \times n}$, $M \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{m \times n}$ be the corresponding matrices. Let us also define the scalar Laplace matrix on Q_h as $L = (L_{i,j})_{i,j=1}^m \in \mathbb{R}^{m \times m}$, where

$$L_{i,j} = \int_\Omega \nabla \phi_j \cdot \nabla \phi_i\,\mathrm{d}x \tag{5}$$

We further introduce the load vector $g \in \mathbb{R}^n$ by setting

$$g_i = \int_\Omega f \cdot \psi_i\,\mathrm{d}x, \quad 1 \leqslant i \leqslant n$$

where f is the source term in (1). We identify finite element functions $u_h \in V_h$ or $p_h \in Q_h$ with their coefficient vectors $u = (u_1, \ldots, u_n)^\mathrm{T} \in \mathbb{R}^n$ and $p = (p_1, \ldots, p_m)^\mathrm{T} \in \mathbb{R}^m$, with respect to the bases (4). The finite element solution of (3) can now be computed by solving the saddle point linear system (2).

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2.2. Properties of the discrete operators

Let us now present a few key properties of the operators, using the well-known discrete Helmholtz decomposition for Nédélec elements. To that end, note that $\nabla Q_h \subset V_h$, and let us introduce the matrix $C \in \mathbb{R}^{n\times m}$ by setting

$$\nabla\phi_j = \sum_{i=1}^{n} C_{i,j}\psi_i, \quad j=1,\ldots,m$$

For a function $q_h \in Q_h$ given by $q_h = \sum_{j=1}^{m} q_j\phi_j$, we then have

$$\nabla q_h = \sum_{i=1}^{n}\sum_{j=1}^{m} C_{i,j} q_j \psi_i$$

so that for $q=(q_1,\ldots,q_m)^{\mathrm{T}}$, we have that $u=Cq$ is the coefficient vector of $u_h=\nabla q_h$ in the basis $\langle\psi_i\rangle_{i=1}^n$.

We shall denote by $\langle\cdot,\cdot\rangle$ the standard Euclidean inner product in $\mathbb{R}^n$ or $\mathbb{R}^m$, and by null$(\cdot)$ the null space of a matrix. For a given positive (semi)definite matrix W and a vector x, we define the (semi)norm

$$|x|_W = \sqrt{\langle Wx, x\rangle}$$

Proposition 2.1
The following relations hold:

(i) $\mathbb{R}^n = \mathrm{null}(A) \oplus \mathrm{null}(B)$.
(ii) For any $u \in \mathrm{null}(A)$ there is a unique $q \in \mathbb{R}^m$ such that $u = Cq$.
(iii) $\langle Mu, Cq\rangle = \langle Bu, q\rangle$ for $u \in \mathbb{R}^n$ and $q \in \mathbb{R}^m$.
(iv) $\langle MCp, Cq\rangle = \langle Lp, q\rangle$ for $p, q \in \mathbb{R}^m$.
(v) Let $u \in \mathrm{null}(A)$ with $u = Cp$. Then $|u|_M = |p|_L$.

Proof
The first two relations readily follow from the discrete Helmholtz decomposition, see, for example, [6, Section 7.2.1]. If u_h and q_h are the finite element functions associated with the vectors u and q, then we have

$$\langle Mu, Cq\rangle = \int_\Omega u_h \cdot \nabla q_h \,\mathrm{d}x = \langle Bu, q\rangle$$

which shows (iii). Relation (iv) follows similarly, and (v) follows from (iv). □

Let us further show a few connections of C to the other matrices.

Proposition 2.2
The following relations hold:

(i) $AC = 0$.
(ii) $BC = L$.
(iii) $MC = B^{\mathrm{T}}$.
(iv) If the datum f is divergence free, then $C^{\mathrm{T}} g = 0$.

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Proof
The first assertion is obvious since the null space of A is equal to the range of C, by Proposition 2.1. The defining properties of B, C and L yield, for $1\leqslant i,\ j\leqslant m$,

$$(BC)_{i,j}=\sum_{k=1}^{n}B_{i,k}C_{k,j}=\int_{\Omega}\left(\sum_{k=1}^{n}C_{k,j}\psi_k\right)\cdot\nabla\phi_i\,\mathrm{d}x=\int_{\Omega}\nabla\phi_j\cdot\nabla\phi_i\,\mathrm{d}x=L_{i,j}$$

This shows identity (ii). The third one follows similarly. Finally, to see (iv), note that for $1\leqslant j\leqslant m$, using integration by parts and the divergence-free condition, we obtain

$$(C^{\mathrm{T}}g)_j=\sum_{i=1}^{n}C_{i,j}g_i=\int_{\Omega}f\cdot\nabla\phi_j\,\mathrm{d}x=-\int_{\Omega}(\nabla\cdot f)\,\phi_j\,\mathrm{d}x=0$$

This completes the proof. □

An orthogonality property with respect to the inner product $\langle M\cdot,\cdot\rangle$ is obtained as follows. Let $u_A\in\mathrm{null}(A)$ and $u_B\in\mathrm{null}(B)$. Setting $u_A=Cq$, we have

$$\langle Mu_A,u_B\rangle=\langle Mu_B,Cq\rangle=\langle Bu_B,q\rangle=0 \tag{6}$$

by relation (iii) in Proposition 2.1. Consequently, we also have the following result.

Proposition 2.3
Let $u=u_A+u_B$ with $u_A\in\mathrm{null}(A)$ and $u_B\in\mathrm{null}(B)$. Then we have $|u|_M^2=|u_A|_M^2+|u_B|_M^2$.

Let us now present stability properties of the matrices A and B. First, by the Cauchy–Schwarz inequality, we obviously have

$$|\langle Au,v\rangle|\leqslant|u|_A|v|_A,\quad u,v\in\mathbb{R}^n$$

A similar continuity property holds for B:

$$|\langle Bv,q\rangle|\leqslant|v|_M|q|_L,\quad v\in\mathbb{R}^n,\quad q\in\mathbb{R}^m \tag{7}$$

Secondly, the matrix A is positive definite on $\mathrm{null}(B)$ and

$$\langle Au,u\rangle\geqslant\alpha(|u|_A^2+|u|_M^2),\quad u\in\mathrm{null}(B) \tag{8}$$

with a stability constant α which is independent of the mesh size and only depends on the shape regularity of the mesh and the approximation order ℓ [7, Theorem 4.7]. Note that, since $\langle Au,u\rangle=|u|_A^2$, we must have $0<\alpha<1$ and then also

$$|u|_A^2\geqslant\bar{\alpha}|u|_M^2,\quad u\in\mathrm{null}(B) \tag{9}$$

with

$$\bar{\alpha}=\frac{\alpha}{1-\alpha} \tag{10}$$

Finally, the matrix B satisfies the discrete inf–sup condition (see [6, p. 179; 7, p. 319])

$$\inf_{0\neq q\in\mathbb{R}^m}\ \sup_{0\neq v\in\mathrm{null}(A)}\frac{\langle Bv,q\rangle}{|v|_M|q|_L}\geqslant 1 \tag{11}$$

The above-stated properties and the theory of mixed finite element methods [16] ensure that (3) is well-posed and the saddle point system is uniquely solvable (provided that the mesh size is sufficiently small). Moreover, it can been shown that asymptotically the method is optimally convergent in the mesh size; see [6, Chapter 7].

3. SPECTRAL EQUIVALENCE PROPERTIES

Consider the augmented matrix

$$A_L = A + B^{\mathrm{T}} L^{-1} B \tag{12}$$

where L is the scalar Laplacian defined in (5). The spectral equivalence properties derived below motivate the preconditioners presented in Section 5.

Applying the discrete Helmholtz decomposition in Proposition 2.1, we have the following result.

Lemma 3.1
Let $u = u_A + u_B$ with $u_A \in \mathrm{null}(A)$ and $u_B \in \mathrm{null}(B)$. Then

$$|Bu|_{L^{-1}} = |u_A|_M$$

Proof
From Proposition 2.1, we have $u_A = Cp$ for a vector $p \in \mathbb{R}^m$. Using the identity $BC = L$ in Proposition 2.2, we obtain

$$|Bu|^2_{L^{-1}} = \langle L^{-1}Bu, Bu\rangle = \langle L^{-1}Bu_A, Bu_A\rangle = \langle L^{-1}BCp, BCp\rangle = \langle Lp, p\rangle = |p|^2_L$$

Since $|p|_L = |u_A|_M$, the result follows. □

As an immediate consequence of Lemma 3.1, we conclude that $B^{\mathrm{T}}L^{-1}B$ and M are spectrally equivalent on the null space of A.

Corollary 3.2
For any u in the null space of A the following relation holds:

$$\langle B^{\mathrm{T}}L^{-1}Bu, u\rangle = \langle Mu, u\rangle$$

Theorem 3.3
The matrices A_L and $A + M$ are spectrally equivalent:

$$\alpha \leqslant \frac{\langle A_L u, u\rangle}{\langle (A+M)u, u\rangle} \leqslant 1$$

for any $u \in \mathbb{R}^n$, where $0 < \alpha < 1$ is the coercivity constant in (8).

By noticing that

$$\langle (A+M)u, u\rangle = |u|^2_A + |u|^2_M$$

the proof of Theorem 3.3 is readily obtained from the bounds in the subsequent lemma. We note that a similar result can be found in [12, Theorem 3.1].

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Lemma 3.4
The following relations hold:

(i) $|\langle A_L u, v\rangle| \leqslant (|u|_A^2 + |u|_M^2)^{1/2}(|v|_A^2 + |v|_M^2)^{1/2}$ for $u, v \in \mathbb{R}^n$.
(ii) $\langle A_L u, u\rangle \geqslant \alpha(|u|_A^2 + |u|_M^2)$ for $u \in \mathbb{R}^n$.

In (ii) $0 < \alpha < 1$ is the coercivity constant given in (8).

Proof
By Proposition 2.1, we may decompose u and v into $u = u_A + u_B$ and $v = v_A + v_B$ with u_A, $v_A \in \mathrm{null}(A)$ and u_B, $v_B \in \mathrm{null}(B)$. Furthermore, there are vectors p and q in $\mathbb{R}^m$ such that $u_A = Cp$ and $v_A = Cq$.

Let us show the first assertion. By the Cauchy–Schwarz inequality,

$$|\langle Au, v\rangle| \leqslant |u|_A |v|_A$$

Similarly, the Cauchy–Schwarz inequality, Lemma 3.1 and the orthogonality in Proposition 2.3 yield

$$|\langle B^{\mathrm{T}} L^{-1} Bu, v\rangle| = |\langle L^{-1} Bu, Bv\rangle| \leqslant |Bu|_{L^{-1}} |Bv|_{L^{-1}} = |u_A|_M |v_A|_M \leqslant |u|_M |v|_M$$

The first assertion readily follows from summing the last two inequalities and applying again the Cauchy–Schwarz inequality. To show the result in (ii), note that the stability property (8) of the matrix A yields:

$$\langle Au, u\rangle = \langle Au_B, u_B\rangle \geqslant \alpha(|u_B|_A^2 + |u_B|_M^2)$$

From Lemma 3.1,

$$\langle L^{-1} Bu, Bu\rangle = |Bu|_{L^{-1}}^2 = |u_A|_M^2$$

and since $0 < \alpha < 1$ we have

$$\langle A_L u, u\rangle \geqslant \alpha(|u_A|_M^2 + |u_B|_A^2 + |u_B|_M^2)$$

By the orthogonality relation in Proposition 2.3 we have $|u|_M^2 = |u_A|_M^2 + |u_B|_M^2$, from which relation (ii) follows. □

We end this section by pointing out a connection between L and the Schur complement associated with A_L, $S = BA_L^{-1}B^{\mathrm{T}}$. The matrices S and L are spectrally equivalent; we have

$$\alpha \leqslant \frac{\langle Sp, p\rangle}{\langle Lp, p\rangle} \leqslant \alpha^{-1}$$

for any $p \in \mathbb{R}^m$. Here, α is the coercivity constant from (8). The proof is a consequence of Lemma 3.4, the inf–sup condition in (11), and standard arguments for mixed finite element methods [16]. We also refer the reader to [12, Theorem 3.3]. As a consequence, the preconditioners we propose in the sequel are closely related to block preconditioners that rely on forming approximations of the Schur complement. Such techniques have been successfully used in a variety of applications, notably for the discretized Stokes and Navier–Stokes equations [17, 18].

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4. AUGMENTATION WITH THE SCALAR LAPLACIAN

We now turn our attention to the linear system and consider augmentation with the Laplacian as a starting point. We will assume that $A - k^2 M$ is non-singular; this can always be achieved by choosing the mesh size sufficiently small [6, Corollary 7.3].

Consider the matrix of (2):

$$\mathscr{K} = \begin{pmatrix} A - k^2 M & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \tag{13}$$

and define the symmetric positive definite block diagonal matrix

$$\mathscr{K}_L = \begin{pmatrix} A_L - k^2 M & 0 \\ 0 & L \end{pmatrix} \tag{14}$$

We stress that $\mathscr{K}_L$ will not be the preconditioner that we eventually use; it is only introduced to lay the theoretical basis and motivation for the preconditioning approach that we propose in Section 5. Note that $A_L - k^2 M$ is symmetric positive definite for k sufficiently small.

Theorem 4.1
The matrix $\mathscr{K}_L^{-1}\mathscr{K}$ has two distinct eigenvalues, given by

$$\mu_+ = 1, \quad \mu_- = -\frac{1}{1-k^2}$$

with algebraic multiplicities n and m, respectively.

Proof
The matrix $\mathscr{K}_L^{-1}\mathscr{K}$ has a complete set of linearly independent eigenvectors that span $\mathbb{R}^{n+m}$. The corresponding eigenvalue problem is

$$\begin{pmatrix} A - k^2 M & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \mu \begin{pmatrix} A - k^2 M + B^{\mathrm{T}} L^{-1} B & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix}$$

From the non-singularity of $\mathscr{K}_L^{-1}\mathscr{K}$ it follows that $\mu \neq 0$. Substituting $q = \mu^{-1} L^{-1} B v$, we obtain for the first block row

$$\mu(A - k^2 M)v + B^{\mathrm{T}} L^{-1} B v = \mu^2 (A - k^2 M + B^{\mathrm{T}} L^{-1} B) v \tag{15}$$

By inspection it is straightforward to see that any vector $v \in \mathbb{R}^n$ satisfies (15) with $\mu = 1$, and thus the latter is an eigenvalue of $\mathscr{K}_L^{-1}\mathscr{K}$, with eigenvectors of the form $(v, L^{-1}Bv)$, where $v \neq 0$. We claim that the eigenvalue $\mu = 1$ has algebraic multiplicity n. (That is, there is no other eigenvector associated with $\mu = 1$ in addition to the above set.) Indeed, if the vectors $\{(v_i, L^{-1}Bv_i)\}_{i=1}^{n+r}$, with $r \geqslant 0$, are linearly independent then necessarily $\{v_i\}_{i=1}^{n+r}$ are also linearly independent, and the latter is impossible unless $r = 0$.

Let us now point out a specific set of eigenvectors for $\mu = 1$, and derive expressions for the remaining eigenpairs. According to Proposition 2.1 we can decompose $v = v_A + v_B$, where $v_A \in \operatorname{null}(A)$ and $v_B \in \operatorname{null}(B)$. We now show that if an eigenvector $(v, \mu^{-1}L^{-1}Bv) = (v_A +$

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$v_B, \mu^{-1}L^{-1}Bv_A)$ has a non-zero v_B component, then its associated eigenvalue must necessarily be $\mu = 1$. Noting that by (6)

$$\langle M(v_A + v_B), v_B \rangle = |v_B|_M^2$$

after taking inner products of (15) with v_B and dividing by μ we get

$$(\mu - 1)(|v_B|_A^2 - k^2|v_B|_M^2) = 0$$

Since the symmetric matrix $A - k^2M$ is non-singular, it follows that for $v_B \neq 0$ we must have $|v_B|_A^2 - k^2|v_B|_M^2 = \langle (A - k^2M)v_B, v_B \rangle \neq 0$, and hence $\mu = 1$.

Next, we argue that at least $2m$ of the vectors v must have a non-zero v_A component. Let us prove this by showing that assuming otherwise leads to a contradiction. Suppose the eigenvectors are given by $(v, \mu^{-1}L^{-1}Bv)$ for a set of $n + m$ choices of v. If the assumption does not hold, then more than $n - m$ eigenvectors satisfy $v = v_B$, and must be of the form $(v_B, 0)$. But since the null space of B is of rank $n - m$, there cannot be more than this number of linearly independent vectors $(v_B, 0)$.

Since at least $2m$ of the eigenvectors satisfy $v_A \neq 0$, and since the multiplicity of $\mu = 1$ is n, it follows that at least m of the eigenvectors associated with $\mu = 1$ satisfy $v_A \neq 0$. Thus, consider m such vectors, $v = v_A + v_B$ with $v_A \neq 0$. Then (15) reads

$$\mu(Av_B - k^2M(v_A + v_B)) + B^{\mathrm{T}}L^{-1}Bv_A = \mu^2(Av_B - k^2M(v_A + v_B) + B^{\mathrm{T}}L^{-1}Bv_A)$$

Taking inner products with the vectors v_A and noting that by (6)

$$\langle M(v_A + v_B), v_A \rangle = |v_A|_M^2$$

and by Corollary 3.2 we have

$$\langle B^{\mathrm{T}}L^{-1}Bv_A, v_A \rangle = \langle Mv_A, v_A \rangle$$

it follows that

$$-(\mu^2 - \mu)k^2|v_A|_M^2 + (\mu^2 - 1)|v_A|_M^2 = 0$$

Hence we have

$$(1 - k^2)\mu^2 + k^2\mu - 1 = 0 \tag{16}$$

from which it follows that $\mu_+ = 1$ and $\mu_- = -1/(1 - k^2)$. We have thus shown that μ_- is the only possible eigenvalue that is not equal to 1, and its algebraic multiplicity must be equal to m. This completes the proof. □

The proof of Theorem 4.1 in fact shows that the eigenspace of $\mathscr{K}_L^{-1}\mathscr{K}$ can be expressed in terms of the null vectors of A and B, as follows.

Corollary 4.2
Let $\{v_i\}_{i=1}^m$ be a basis for the null space of A and $\{z_i\}_{i=1}^{n-m}$ a basis for the null space of B. Then $\{(v_i, L^{-1}Bv_i)\}_{i=1}^m$ and $\{(z_i, 0)\}_{i=1}^{n-m}$ are n linearly independent eigenvectors associated with the eigenvalue μ_+. The vectors $\{(v_i, -(1-k^2)L^{-1}Bv_i)\}_{i=1}^m$ are m linearly independent eigenvectors

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associated with the eigenvalue μ_-. Grouped together, those eigenvectors form a complete eigenspace that spans $\mathbb{R}^{n+m}$.

From Theorem 4.1 it follows that if MINRES were to be used for solving (2), with $\mathscr{K}_L$ as a preconditioner, then convergence would require merely two iterations, if roundoff errors are ignored. However, forming A_L may be too computationally costly. We mention also that the results in Theorem 5.2 can be extended to general algebraic settings, i.e. not necessarily to the Maxwell operator, as we show in [19].

5. AUGMENTATION-FREE PRECONDITIONERS

Define

$$\mathscr{P}_M = A + \gamma M \tag{17}$$

where $\gamma = 1 - k^2$. For the saddle point system (2) we consider preconditioning with

$$\mathscr{P}_{M,L} = \begin{pmatrix} \mathscr{P}_M & 0 \\ 0 & L \end{pmatrix} \tag{18}$$

Throughout, we will assume that preconditioned MINRES for the saddle point system is used. A crucial factor in the speed of convergence of this method is the distribution of the eigenvalues; strong clustering yields fast convergence [20, Section 3.1]. The choice of $\mathscr{P}_M$ and $\mathscr{P}_{M,L}$ is motivated by the spectral equivalence results given in Theorem 3.3 and the eigenvalue distribution observed in Theorem 4.1, which allow us to observe that $\mathscr{P}_M \approx A_L - k^2 M$ and $\mathscr{P}_{M,L} \approx \mathscr{K}_L$. Thus, the overall computational cost of the solution procedure will depend on the ability to efficiently solve linear systems whose associated matrices are $A + \gamma M$ and L (or approximations thereof). For solving the former we refer the reader to [10, 13–15].

Theorem 5.1
The matrix

$$\mathscr{P}_M^{-1}(A_L - k^2 M)$$

has an eigenvalue $\mu = 1$ of algebraic multiplicity m. The rest of the eigenvalues are bounded as follows:

$$\frac{\bar{\alpha} - k^2}{\bar{\alpha} + 1 - k^2} < \mu < 1 \tag{19}$$

with $\bar{\alpha}$ defined in (10).

Proof
The corresponding eigenvalue problem is

$$(A - k^2 M + B^{\mathrm{T}} L^{-1} B)v = \mu(A + (1 - k^2)M)v$$

Suppose $v = v_A + v_B$, where $v_A \in \mathrm{null}(A)$ and $v_B \in \mathrm{null}(B)$. We then have

$$Av_B - k^2 M(v_A + v_B) + B^{\mathrm{T}} L^{-1} B v_A = \mu(Av_B + (1 - k^2)M(v_A + v_B))$$

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By linear independence considerations, there are at least m vectors v that satisfy $v_A \neq 0$. For m such vectors, taking inner products with v_A and noting that by Corollary 3.2

$$\langle B^{\mathrm{T}} L^{-1} B v_A, v_A \rangle = |v_A|_M^2$$

and that by (6) we have

$$\langle M(v_A + v_B), v_A \rangle = \langle M v_A, v_A \rangle = |v_A|_M^2$$

we get

$$\mu(1-k^2)|v_A|_M^2 = (1-k^2)|v_A|_M^2$$

It follows that $\mu = 1$ is an eigenvalue of multiplicity m.

For the rest of the eigenvectors we must have $v_B \neq 0$, and now taking inner products with v_B and noting that

$$\langle B^{\mathrm{T}} L^{-1} B v_A, v_B \rangle = \langle L^{-1} B v_A, B v_B \rangle = 0$$

and that by (6) we have

$$\langle M(v_A + v_B), v_B \rangle = \langle M v_B, v_B \rangle = |v_B|_M^2$$

it follows that

$$(1-\mu)|v_B|_A^2 = \left((1-k^2)\mu + k^2\right)|v_B|_M^2 \tag{20}$$

It is impossible to have $\mu = 1$, since in this case (20) collapses into $|v_B|_M = 0$, which cannot hold for $v_B \neq 0$. We cannot have $\mu > 1$ either, since that would imply that in (20) the left-hand side is negative but the right-hand side is positive. (Recall that we assume $k \ll 1$.) We conclude that we must have $\mu < 1$.

From (9) we recall that for any $u \in \mathrm{null}(B)$, $|u|_A^2 \geqslant \bar{\alpha}|u|_M^2$ with $\bar{\alpha} = \alpha/(1-\alpha) > 0$. Applying this to (20) we conclude $(1-k^2)\mu + k^2 \geqslant \bar{\alpha}(1-\mu)$, and since $1 - k^2 + \bar{\alpha} > 0$ we obtain (19). Since μ can be either equal to 1 or satisfy (19), but not simultaneously both, the algebraic multiplicities follow. $\square$

Theorem 5.2

Let $\mathscr{K}$ be the saddle point matrix (13). Then $\mu_+ = 1$ and $\mu_- = -1/(1-k^2)$ are eigenvalues of the preconditioned matrix $\mathscr{P}_{M,L}^{-1}\mathscr{K}$, each with algebraic multiplicity m. The rest of the eigenvalues satisfy the bound (19).

Proof

The eigenvalue problem for $\mathscr{P}_{M,L}^{-1}\mathscr{K}$ is

$$\begin{pmatrix} A - k^2 M & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix} = \mu \begin{pmatrix} A + (1-k^2)M & 0 \\ 0 & L \end{pmatrix} \begin{pmatrix} v \\ q \end{pmatrix}$$

Setting $q = \mu^{-1} L^{-1} B v$ and multiplying the resulting equation for v by μ, we have

$$[(\mu^2 - \mu)A + ((1-k^2)\mu^2 + k^2\mu)M]v = B^{\mathrm{T}} L^{-1} B v$$

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The rest of the proof follows by taking the same steps taken in the proof of Theorem 5.1. We get m equations of the form

$$(1-k^2)\mu^2 + k^2\mu - 1 = 0$$

from which μ_+ and μ_- are obtained. Note that this quadratic equation is identical to Equation (16) for the eigenvalues of $\mathscr{K}_L^{-1}\mathscr{K}$, cf. Theorem 4.1. This reinforces that $\mathscr{P}_{M,L}$ is an effective sparse approximation of $\mathscr{K}_L$. Obtaining the bound (19) is done in a way identical to the last part of the proof of Theorem 5.1. □

For $k=0$ the result of Theorem 5.2 simplifies as follows.

Corollary 5.3

For the preconditioned matrix $\mathscr{P}_{M,L}^{-1}\mathscr{K}$ with $k=0$, there are eigenvalues $\mu_\pm = \pm 1$, with algebraic multiplicity m each. The rest of the eigenvalues satisfy $\alpha < \mu < 1$.

6. NUMERICAL EXPERIMENTS

Our numerical experiments were performed using MATLAB; for generating the meshes we used the PDE toolbox. We implemented the two-dimensional version of the time-harmonic Maxwell equations. The lowest order elements were used, i.e. $\ell = 1$. The solutions of the preconditioned systems in each iteration were computed exactly.

6.1. A smooth domain with a quasi-uniform grid

In this example the domain is the unit square. Uniformly refined meshes were constructed, by subsequently dividing each triangle into four congruent ones. The number of elements and matrix sizes are given in Table I.

First, we set the right-hand side function so that the exact solution is given by

$$u(x,y) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} 1-y^2 \\ 1-x^2 \end{pmatrix}$$

Table I. Number of elements (Nel) and the size of the linear systems ($n+m$) for seven grids used in Example 6.1.

Grid	Nel	$n+m$
G1	64	113
G2	256	481
G3	1024	1985
G4	4096	8065
G5	16 384	32 513
G6	65 536	130 561
G7	262 144	523 265

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Table II. Iteration counts for Example 6.1 with a divergence-free right-hand side, for various meshes and values of k, using MINRES for solving the saddle point system with the preconditioner $\mathscr{P}_{M,L}$.

Grid	$k=0$	$k=\frac{1}{8}$	$k=\frac{1}{4}$	$k=\frac{1}{2}$
G1	5	5	5	5
G2	5	5	5	5
G3	5	5	5	5
G4	6	6	5	6
G5	6	6	6	6
G6	6	6	6	6
G7	6	6	6	6

Note: The outer iteration was stopped once the initial relative residual was reduced by a factor of 10^{-10}.

Table III. Iteration counts for Example 6.1 with a right-hand side that is not divergence free, for various meshes and values of k, using MINRES for solving the saddle point system with the preconditioner $\mathscr{P}_{M,L}$.

Grid	$k=0$	$k=\frac{1}{8}$	$k=\frac{1}{4}$	$k=\frac{1}{2}$
G1	5	5	5	5
G2	6	6	6	6
G3	6	6	6	6
G4	6	6	6	7
G5	7	7	7	7
G6	7	7	7	7
G7	7	7	7	7

Note: The outer iteration was stopped once the initial relative residual was reduced by a factor of 10^{-10}.

and $p \equiv 0$. The datum f in this case is divergence free. We ran MINRES with the preconditioner $\mathscr{P}_{M,L}$. The counts of the outer iterations are given in Table II. The inner iterations were solved by the conjugate gradient method, preconditioned with incomplete Cholesky factorization, using a tight convergence tolerance. As expected, the outer solver scales extremely well with hardly any sensitivity to the mesh size and the wave number.

We also ran the saddle point solver on an example with a right-hand side function that was not divergence free. We took the same u as above, and $p=(1-x^2)(1-y^2)$. The iteration counts are given in Table III. As before, the solver scales very well. Figure 1 depicts the eigenvalues of the preconditioned matrix $\mathscr{P}_{M,L}^{-1}\mathscr{K}$ for grid G2 with $k=\frac{1}{4}$. This linear system has 481 degrees of freedom, with $n=368$ and $m=113$. As is expected from Theorem 5.2, the m negative eigenvalues of the matrix are equal to $-1/(1-k^2)=-\frac{16}{15}=-1.0666\ldots$, and for the positive ones, m of them are equal to 1 and the remaining $n-m$ eigenvalues are bounded away from 0 and below 1. In our computations we observed strong clustering beyond what can be concluded from Theorem 5.2. Three of the positive eigenvalues are between 0.7 and 0.9, with the smallest equal to $0.706\ldots$,

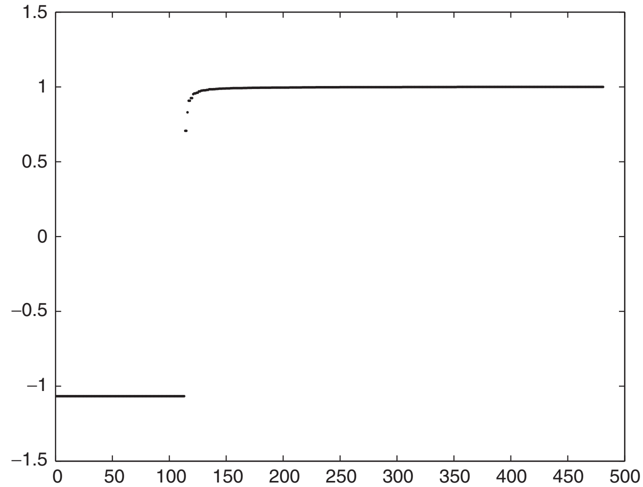

Figure 1. Plot of the eigenvalues of the preconditioned matrix $\mathscr{P}_{M,L}^{-1}\mathscr{K}$, for $k=\frac{1}{4}$, for grid G2 in Example 6.1.

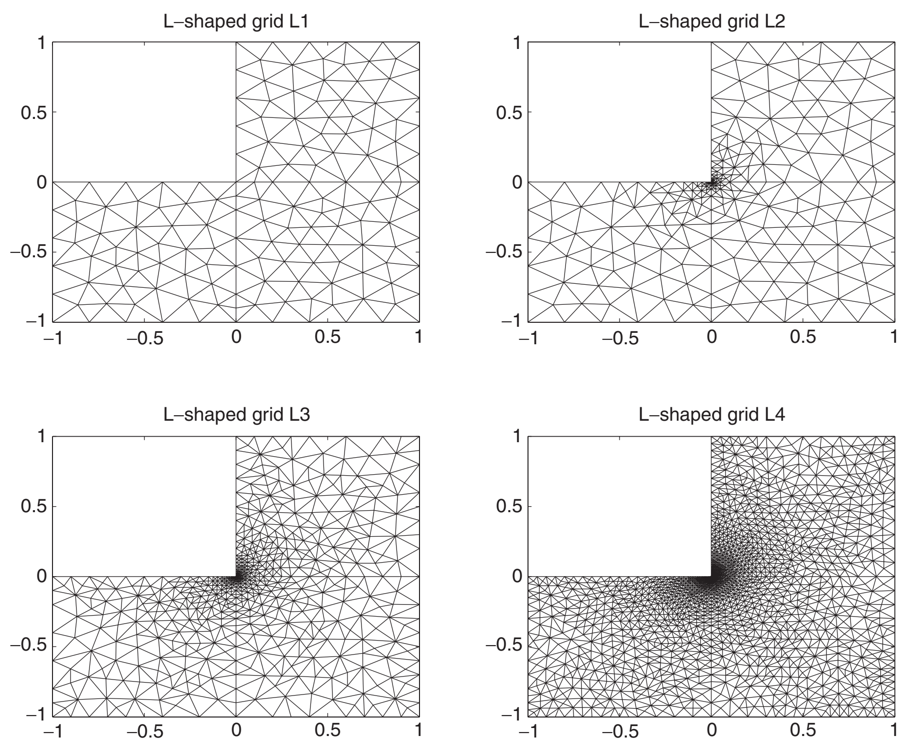

Figure 2. Grids L1–L4 for Example 6.2.

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Table IV. Number of elements (Nel) and the size of the linear systems $(n + m)$ for five grids used in Example 6.2.

Grid	Nel	$n+m$
L1	258	451
L2	458	813
L3	1403	2608
L4	5164	9927
L5	19 339	37 882

Table V. Iteration counts for Example 6.2 with various meshes and values of k, using MINRES for solving the saddle point system with the preconditioner $\mathscr{P}_{M,L}$.

Grid	$k=0$	$k=\frac{1}{8}$	$k=\frac{1}{4}$	$k=\frac{1}{2}$
L1	5	5	5	5
L2	5	5	5	5
L3	5	5	5	5
L4	5	5	5	5
L5	4	4	4	4

Note: The outer iteration was stopped once the initial relative residual was reduced by a factor of 10^{-10}.

and four additional ones are between 0.9 and 0.95. The remaining 361 eigenvalues are all between 0.95 and 1, with 113 of them identically equal to 1, again as is known by the same theorem. This clustering effect explains the fast convergence of the preconditioned iterative solver.

6.2. *An L-shaped domain with locally refined grids*

In this example we consider an L-shaped domain. The meshes were locally refined at the non-convex corner at the origin; the number of elements and sizes are given in Table IV. Four of the five grids that were used are depicted in Figure 2. We set up the problem so that the right-hand side function is equal to 1 throughout the domain. As in the previous example, we applied MINRES, preconditioned by $\mathscr{P}_{M,L}$, to the saddle point system. Table V demonstrates the scalability of the solvers: the outer iteration counts do not seem to be sensitive to changes in the mesh size.

7. CONCLUSIONS

We have introduced a new augmentation-free and Schur complement-free block diagonal preconditioning approach for solving the discretized mixed formulation of the time-harmonic Maxwell equations. We have presented a complete spectral analysis, and have shown that the outer iteration counts are hardly sensitive to changes in the mesh size or in small values of the wave number.

We have limited the discussion in this paper to the convergence of the outer iterations, relying on the assumption that robust solution techniques exist for solving $A + \gamma M$. Future research

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will focus on further computational aspects of our solution technique, and we will explore using efficient inner solvers. Finally, we will investigate whether similar preconditioners can be applied to problems in three dimensions and problems with variable coefficients.

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