Bidding in First-Price Auctions

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Equivalence of First-Price and Dutch Auctions

Theorem

First-Price (sealed bid) and Dutch auctions are strategically equivalent.

- In both, a bidder must decide on the amount s/he's willing to pay, conditional on it being the highest bid.
  - Dutch auctions are extensive-form games, but the only thing a winning bidder knows is that all others have not to bid higher
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  - Dutch auctions are extensive-form games, but the only thing a winning bidder knows is that all others have not to bid higher
  - Same as a bidder in a first-price auction.
- So, why are both auction types used?
  - First-price auctions can be held asynchronously.
  - Dutch auctions are fast, and require minimal communication: only one bit needs to be transmitted from the bidders to the auctioneer.

Discussion

- How should bidders bid in these auctions?
  - Bid less than valuation.
  - There's a tradeoff between:
    - probability of winning
    - amount paid upon winning
  - Bidders don't have a dominant strategy.

Analysis

Theorem

In a first-price auction with two risk-neutral bidders whose valuations are IID and drawn from $U(0,1)$, $(\frac{1}{2}v_1, \frac{1}{2}v_2)$ is a Bayes-Nash equilibrium strategy profile.

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Proof.

Assume that bidder 2 bids $\frac{1}{2}v_2$, and bidder 1 bids s_1.
1 wins when $v_2 < 2s_1$, and gains utility $v_1 - s_1$, but loses when $v_2 > 2s_1$ and then gets utility 0: (we can ignore the case where the agents have the same valuation, because this occurs with probability zero).

$$
\begin{aligned}
E[u_1] &= \int_0^{2s_1} (v_1 - s_1) dv_2 + \int_{2s_1}^{1} (0) dv_2 \\
&= (v_1 - s_1) v_2 \Big|_0^{2s_1} \\
&= 2v_1 s_1 - 2s_1^2. \qquad (1)
\end{aligned}
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Proof Continued.

We can find bidder 1's best response to bidder 2's strategy by taking the derivative of (1) and setting it equal to zero:

$$\begin{aligned}
\frac{\partial}{\partial s_1}(2v_1 s_1 - 2s_1^2) &= 0 \\
2v_1 - 4s_1 &= 0 \\
s_1 &= \frac{1}{2}v_1
\end{aligned}$$

Thus when player 2 is bidding half her valuation, player 1's best reply is to bid half his valuation. The calculation of the optimal bid for player 2 is analogous, given the symmetry of the game.

More than two bidders

- Narrow result: two bidders, uniform valuations.
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- proven using a similar argument.
- A broader problem: the proof only *verified* an equilibrium strategy.
  - How do we find the equilibrium?