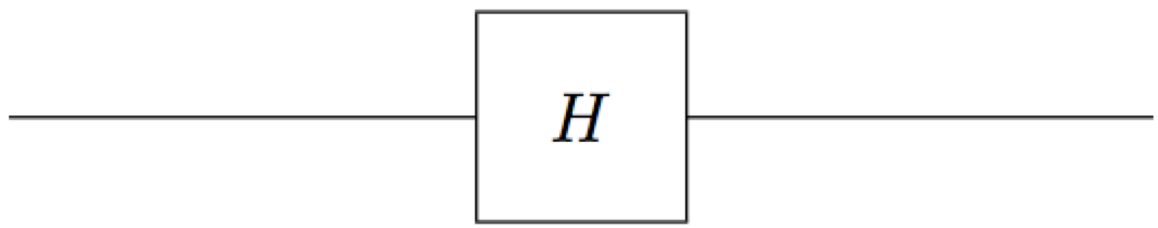
Quantum Circuits

A formal way to represent protocols such as superdense coding, and the basis for defining quantum complexity classes

Slides are based on John Watrous' lectures 3 and 7

Quantum circuits by example

- A Hadamard transform on a single superposition



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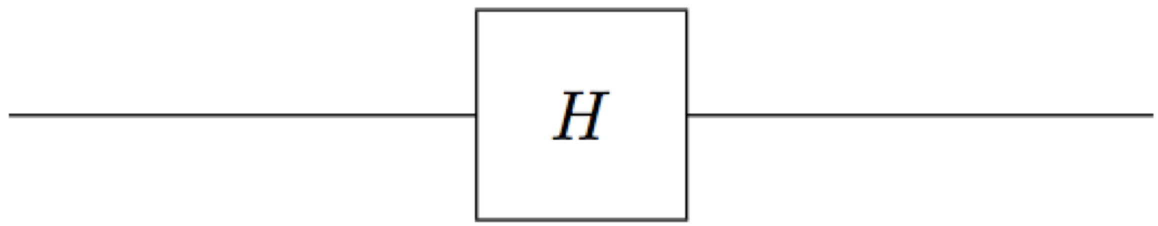
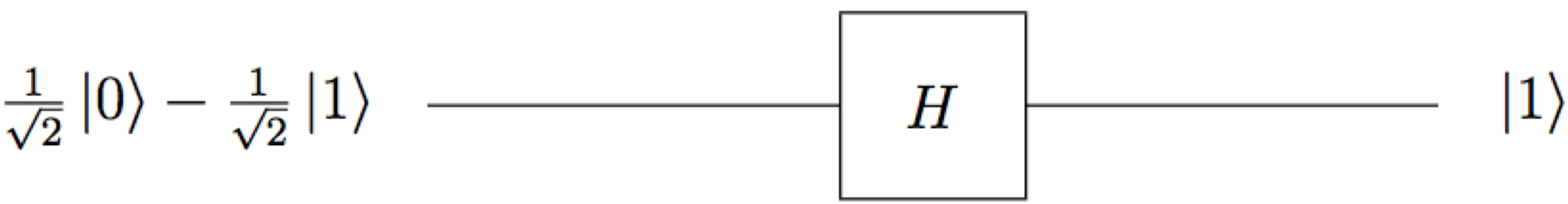


- A Hadamard transform on input $\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$



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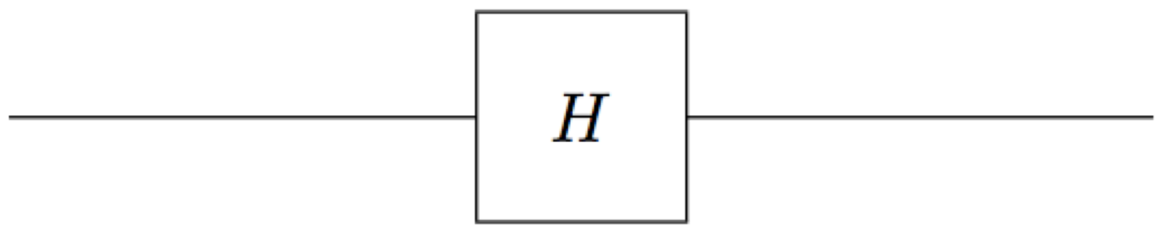

Quantum circuits by example

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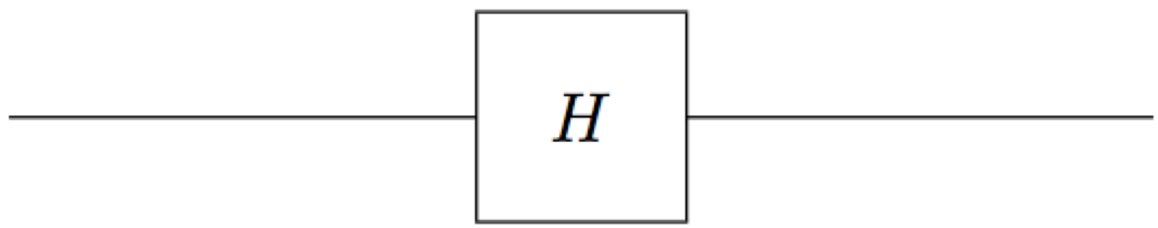
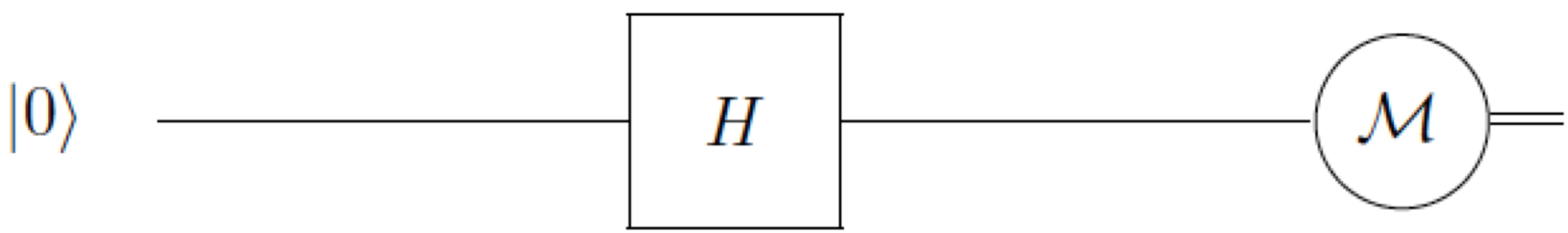
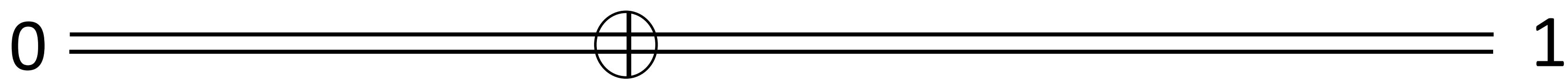


- A measurement produces a classical bit (represented by double lines)



Quantum circuits by example

- A NOT operation with classical input 0



Quantum circuits by example

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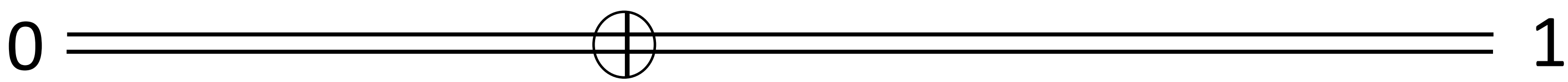
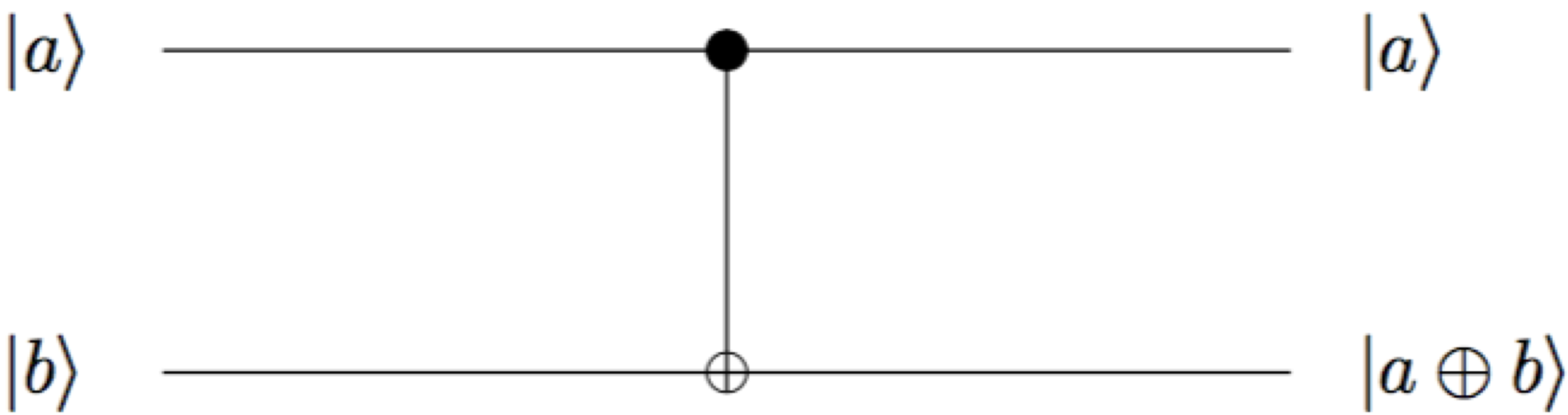
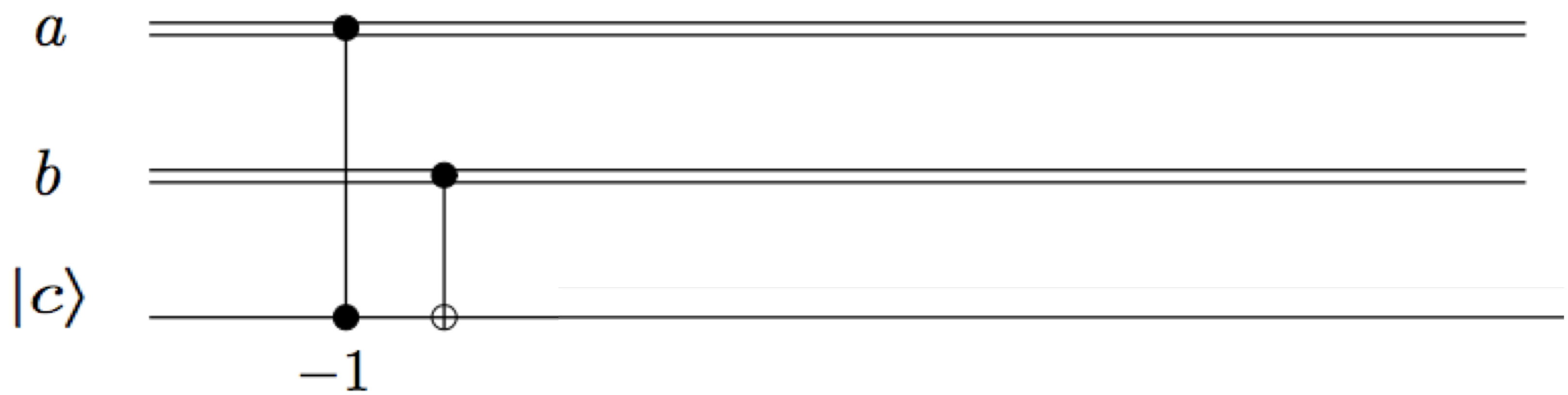


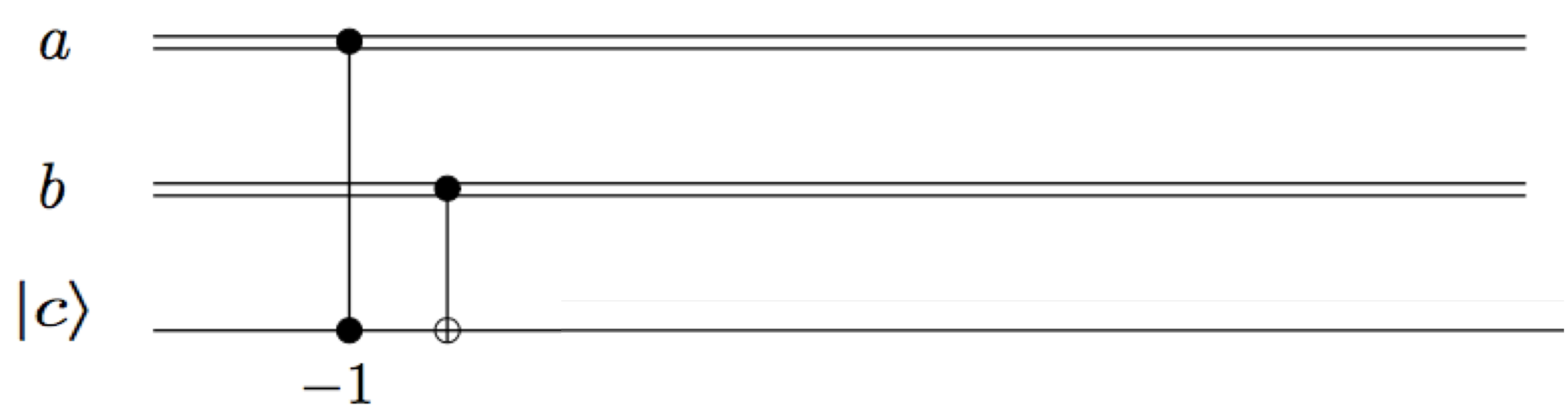
- A controlled NOT operation on inputs (superpositions) *a*, *b* with control *a*



Quantum circuits by example



Quantum circuits by example

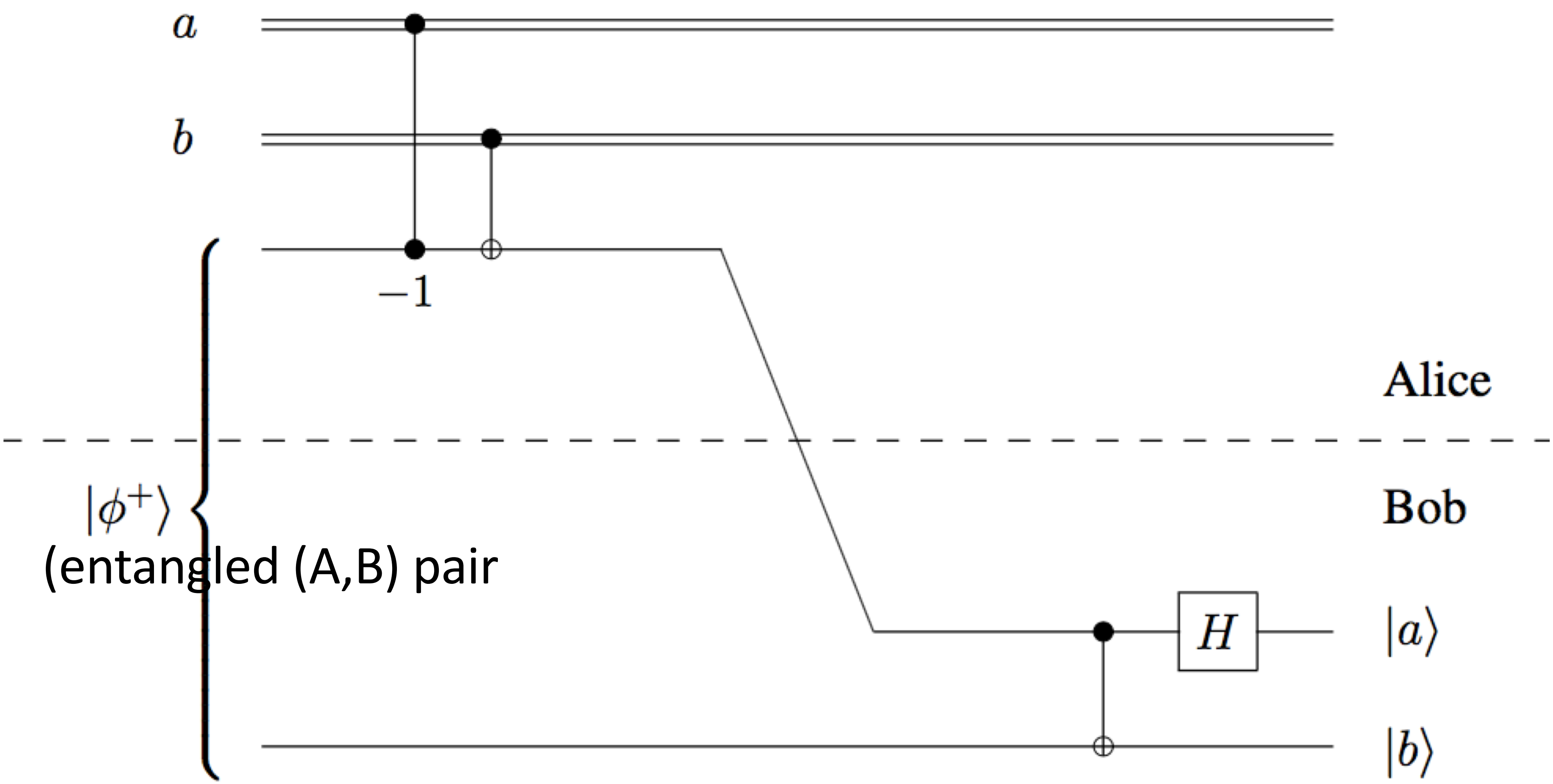


- A controlled $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ operation with control a and target $|c\rangle$
- Then a controlled NOT operation with control b and target which is the output of the first operation
- Here a and b are classical bits

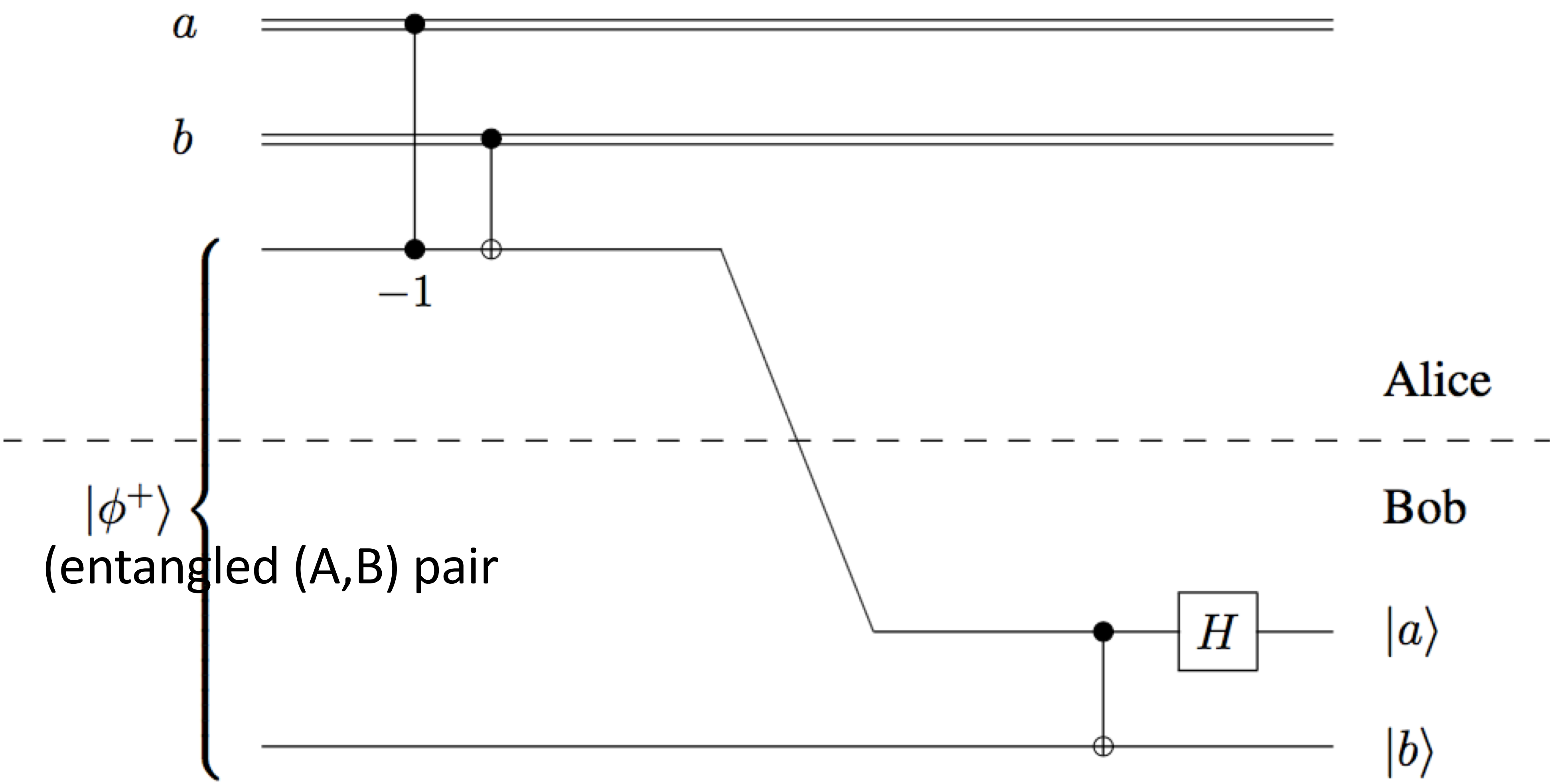
A circuit for superdense coding

1. Alice: If a = 1, apply σ_z to qubit A, where
2. Alice: If b = 1, apply NOT to qubit A
3. Alice: send A to Bob
4. Bob: Apply a controlled-NOT to (A,B) (A is the control)
5. Bob: Apply a Hadamard transform to A
6. Bob: Measure A and B and output the result

A circuit for superdense coding

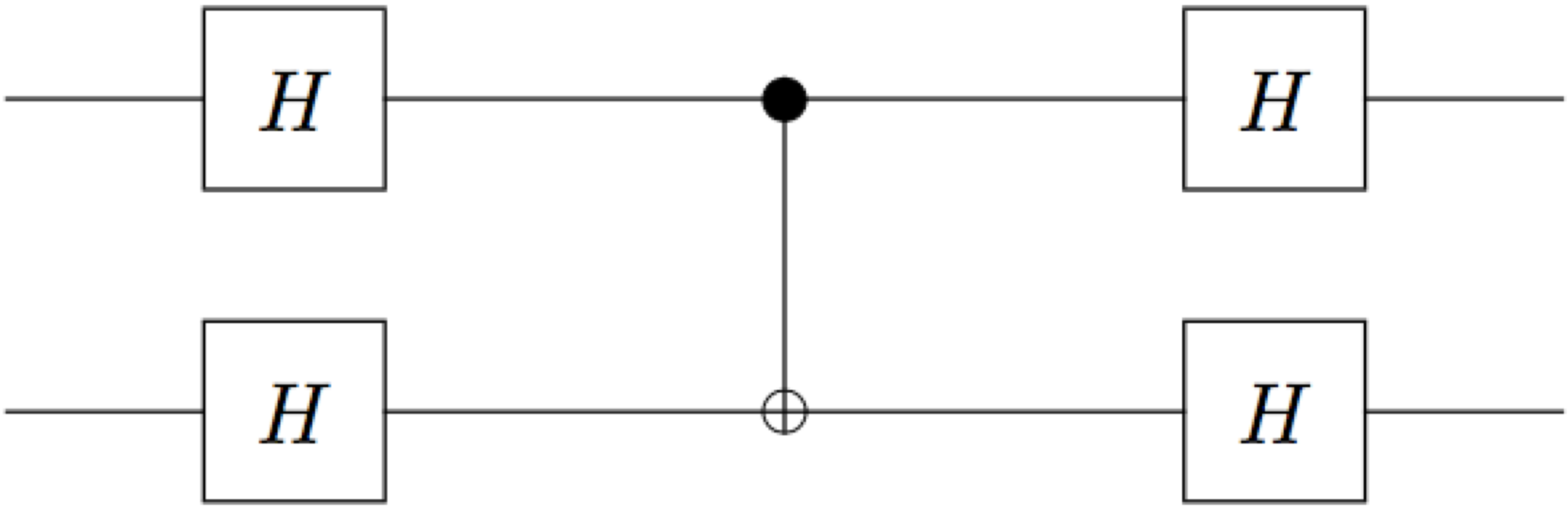


A circuit for superdense coding



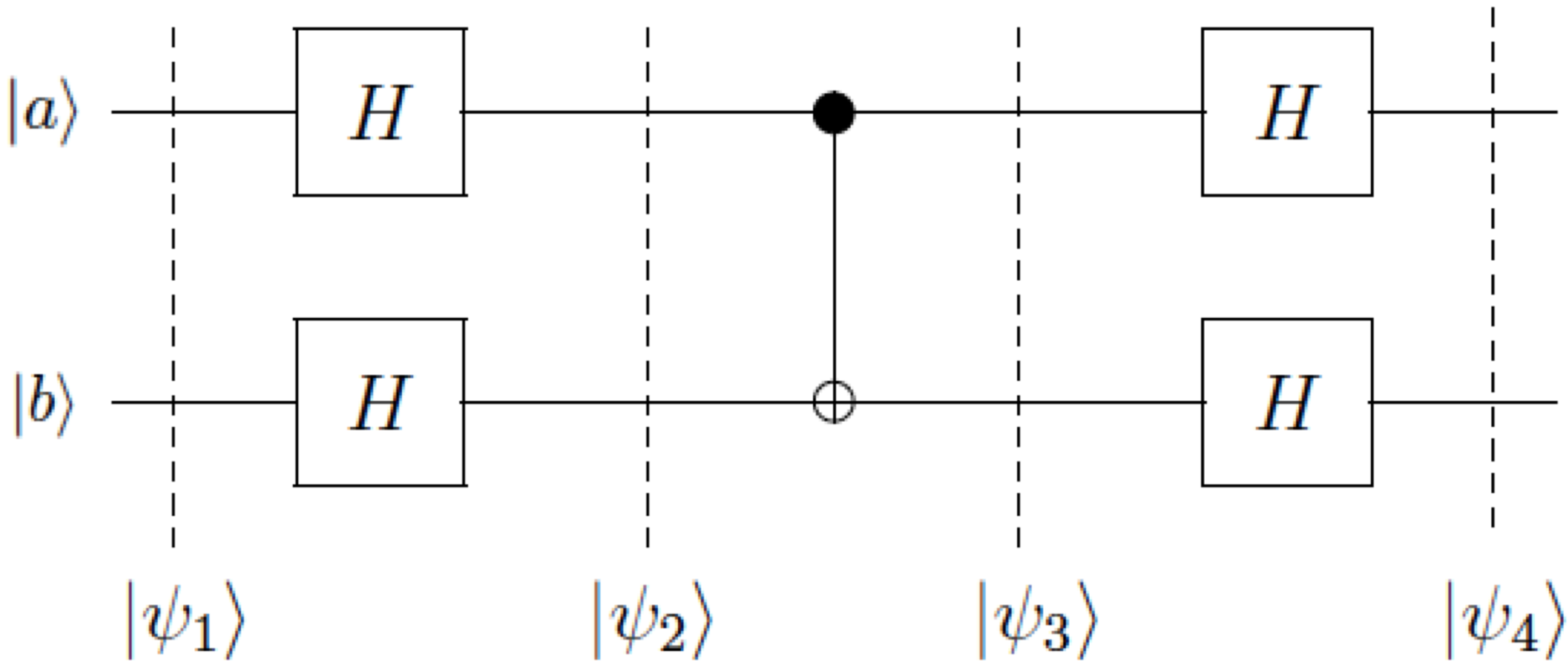
- What do we get if we apply the gates in reverse order to the output?

A circuit with multiple gates



- This circuit has five gates
- Gates are evaluated from left to right
- What do you think that this circuit does?

A circuit with multiple gates



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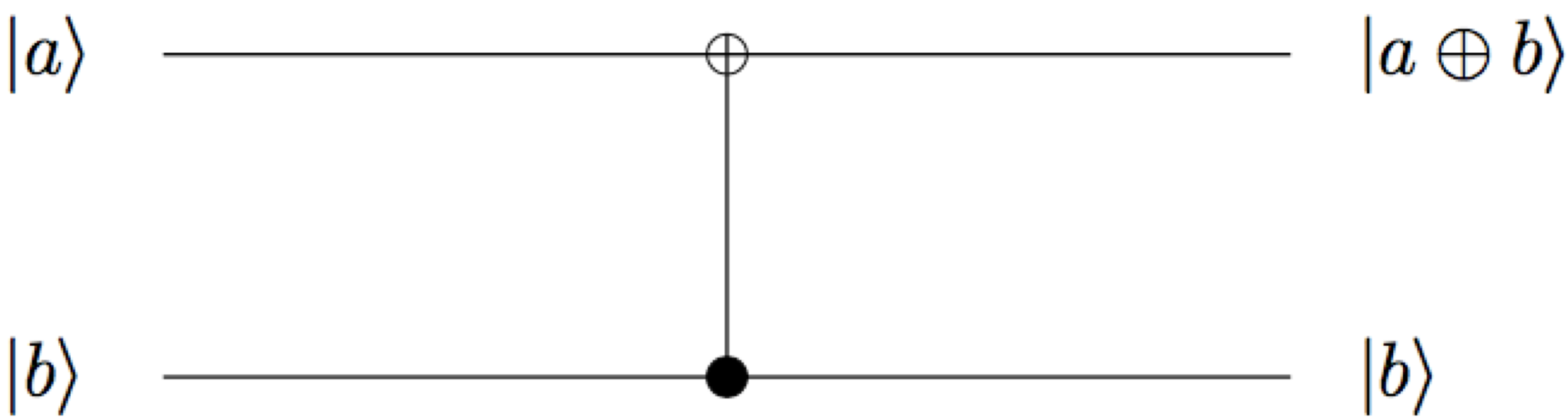
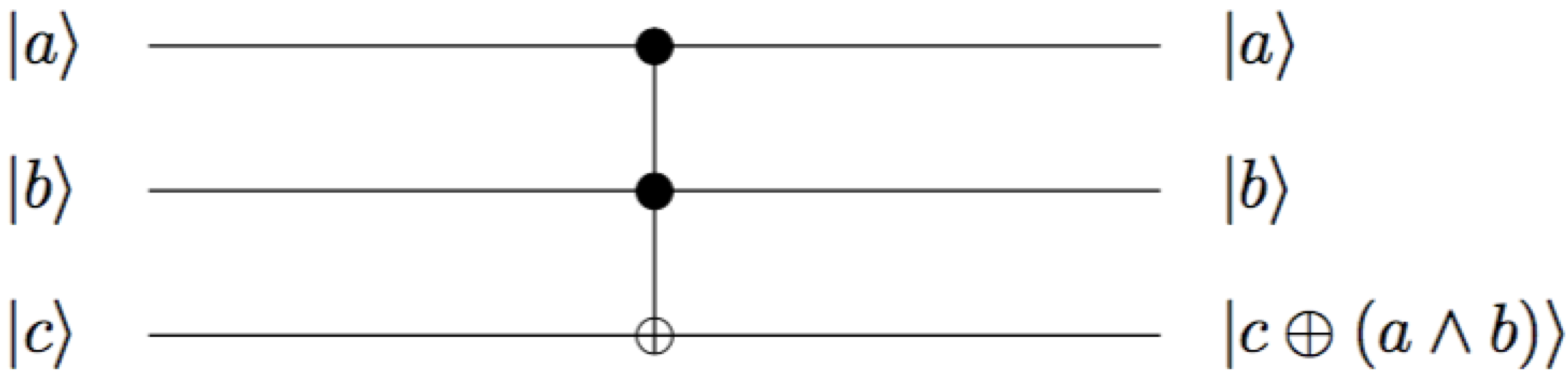
An equivalent circuit



Toffoli gates

- A Toffoli gate is a controlled-controlled NOT operation on inputs *a*, *b, c* in {0,1}; here a and b are the controls



- If c = 0, this gate computes a AND b

Toffoli gates

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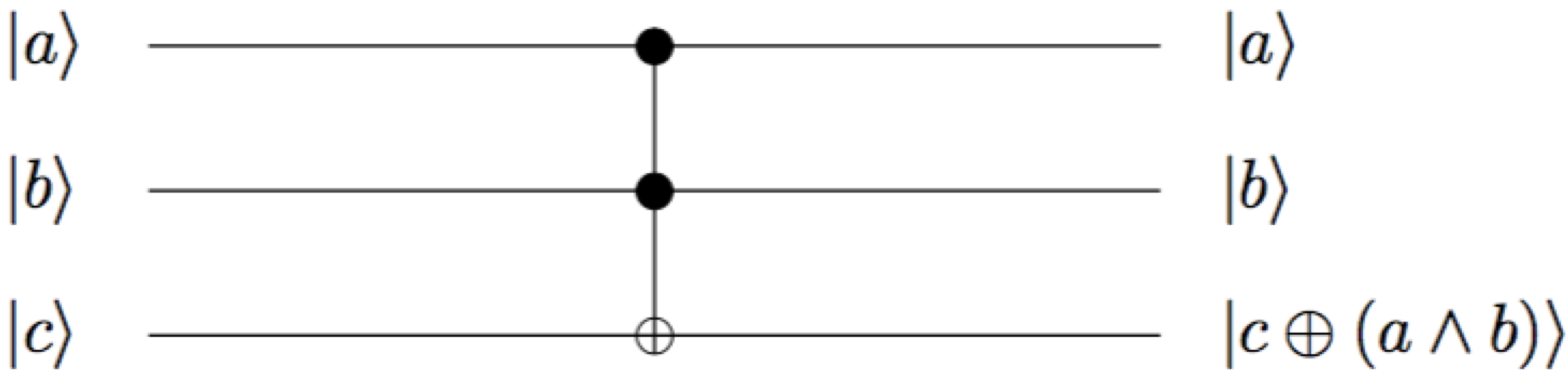


- What is the matrix representing this operation?

Quantum circuit summary

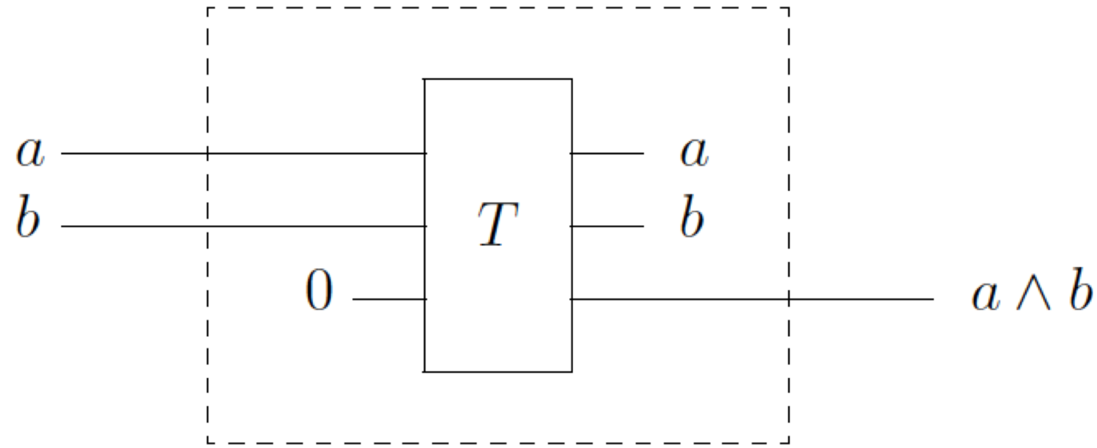
- Qubits are represented as horizontal lines, and the number of input and output bits are equal
- Operations and measurement are represented using various symbols, and are applied from left to right

Simulating classical circuits

- Can any classical circuit be simulated by a quantum circuit?

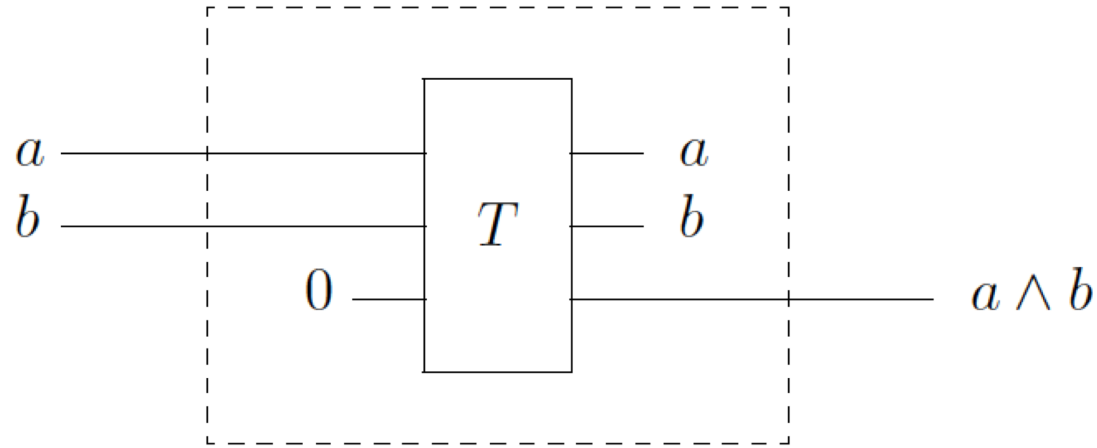
Simulating classical circuits

- AND gates



Simulating classical circuits

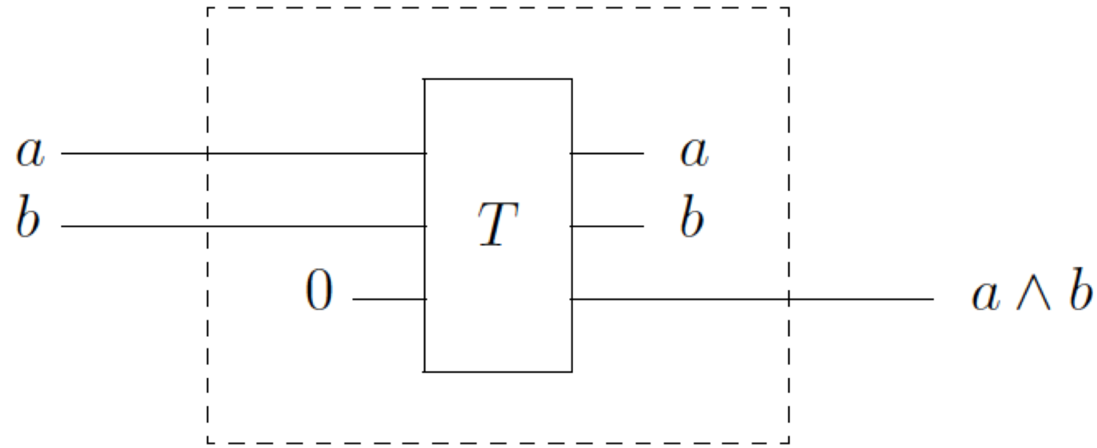
- AND gates



- NOT gates

Simulating classical circuits

- AND gates



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Simulating classical circuits

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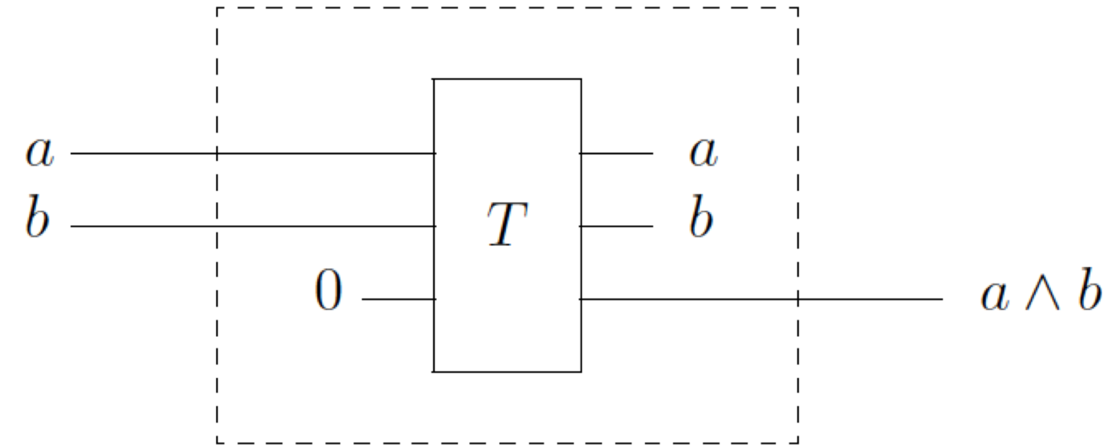
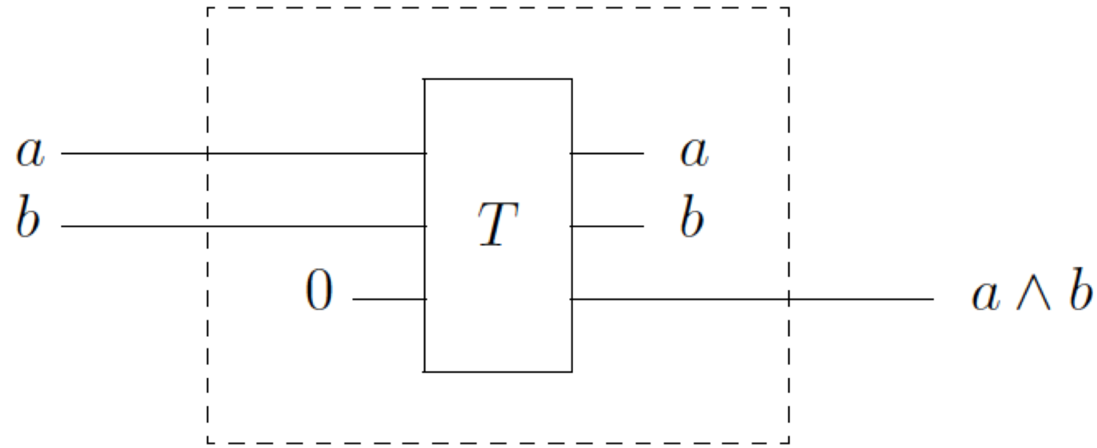


- NOT gates

- NOT can also be simulated by Toffoli – how?

Simulating classical circuits

- AND gates

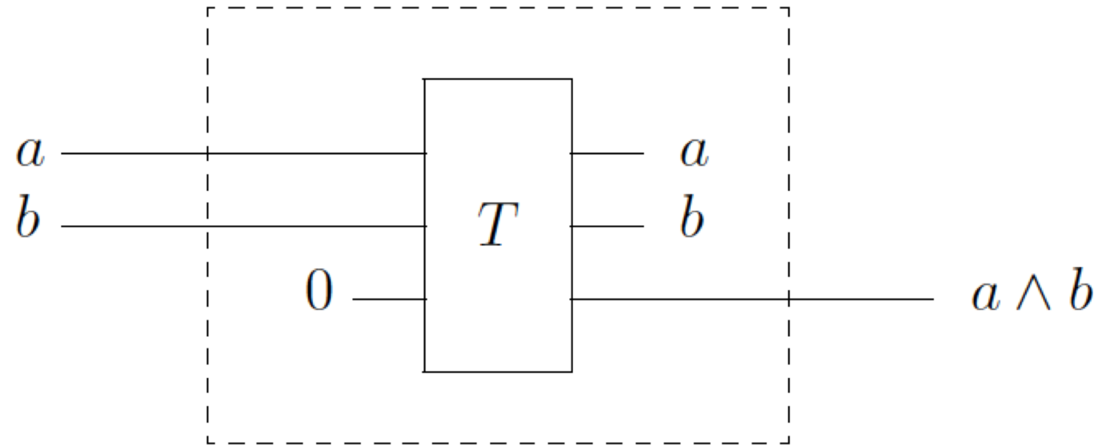


- NOT gates

- Fanout

Simulating classical circuits

- AND gates



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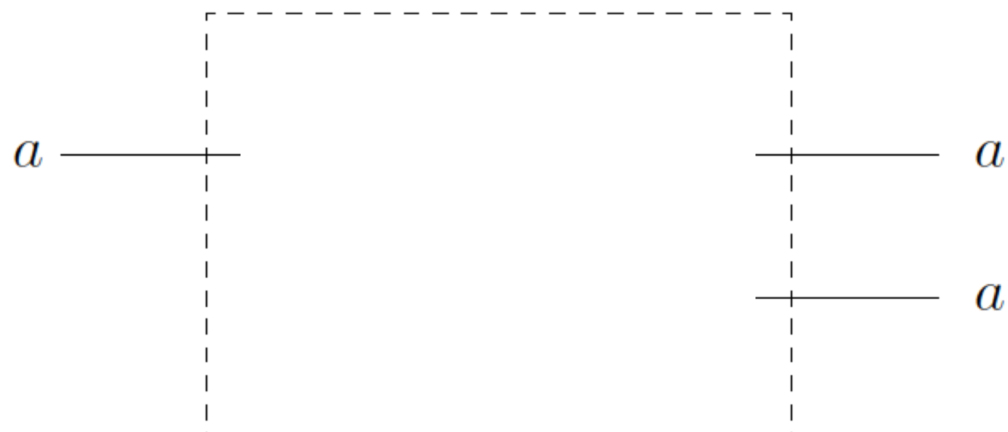

Simulating classical circuits

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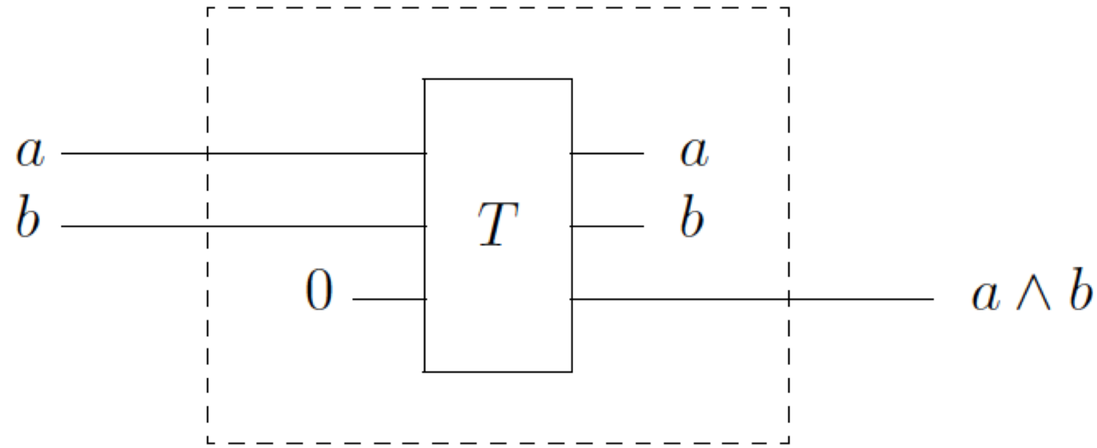

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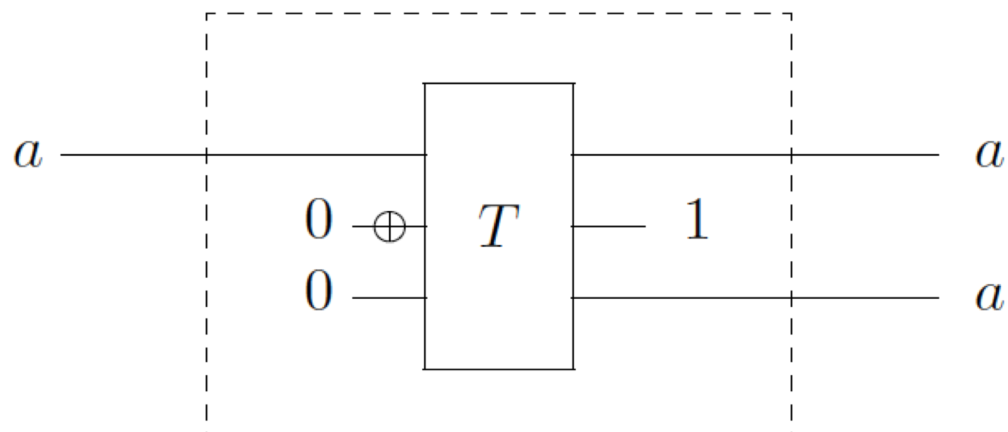

Simulating classical circuits

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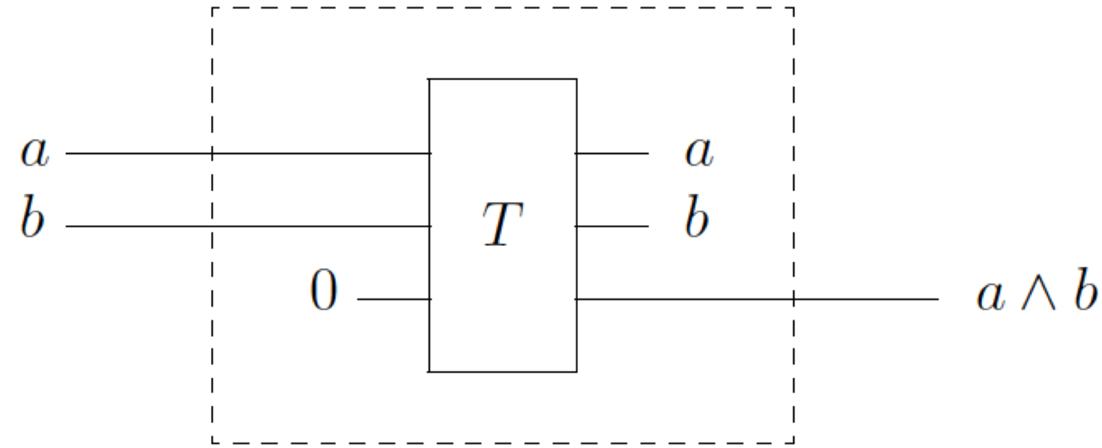

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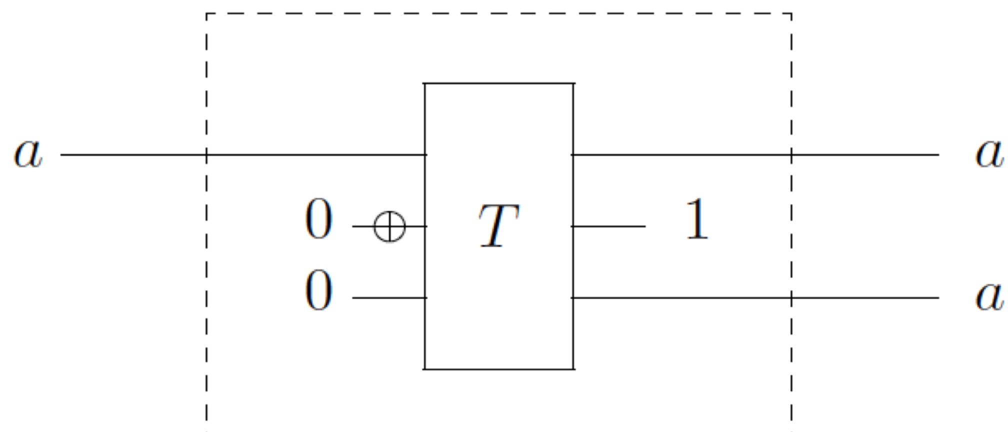
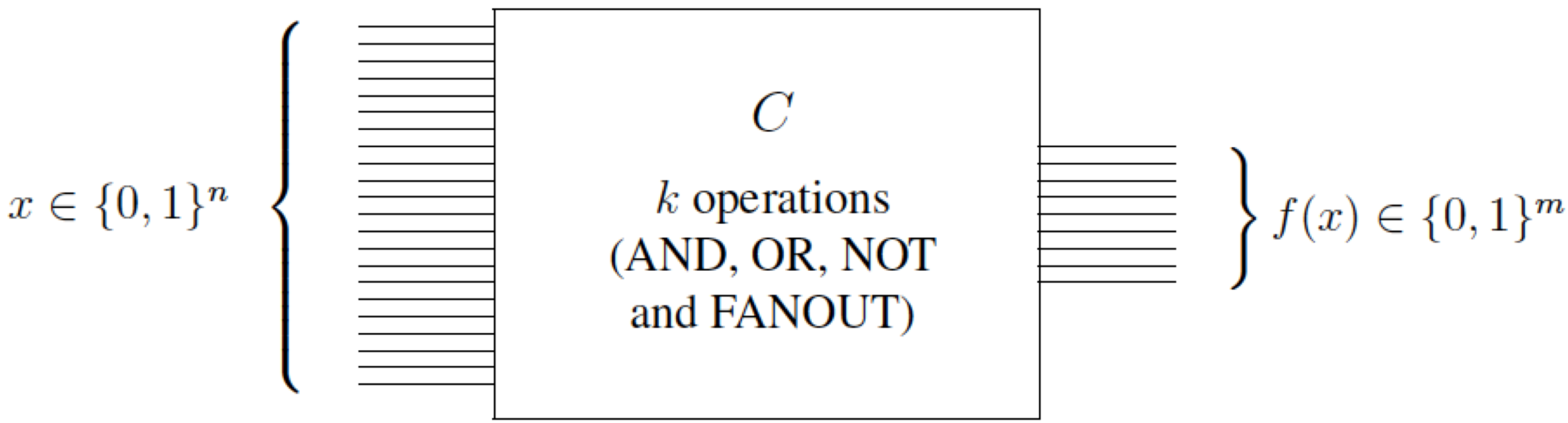


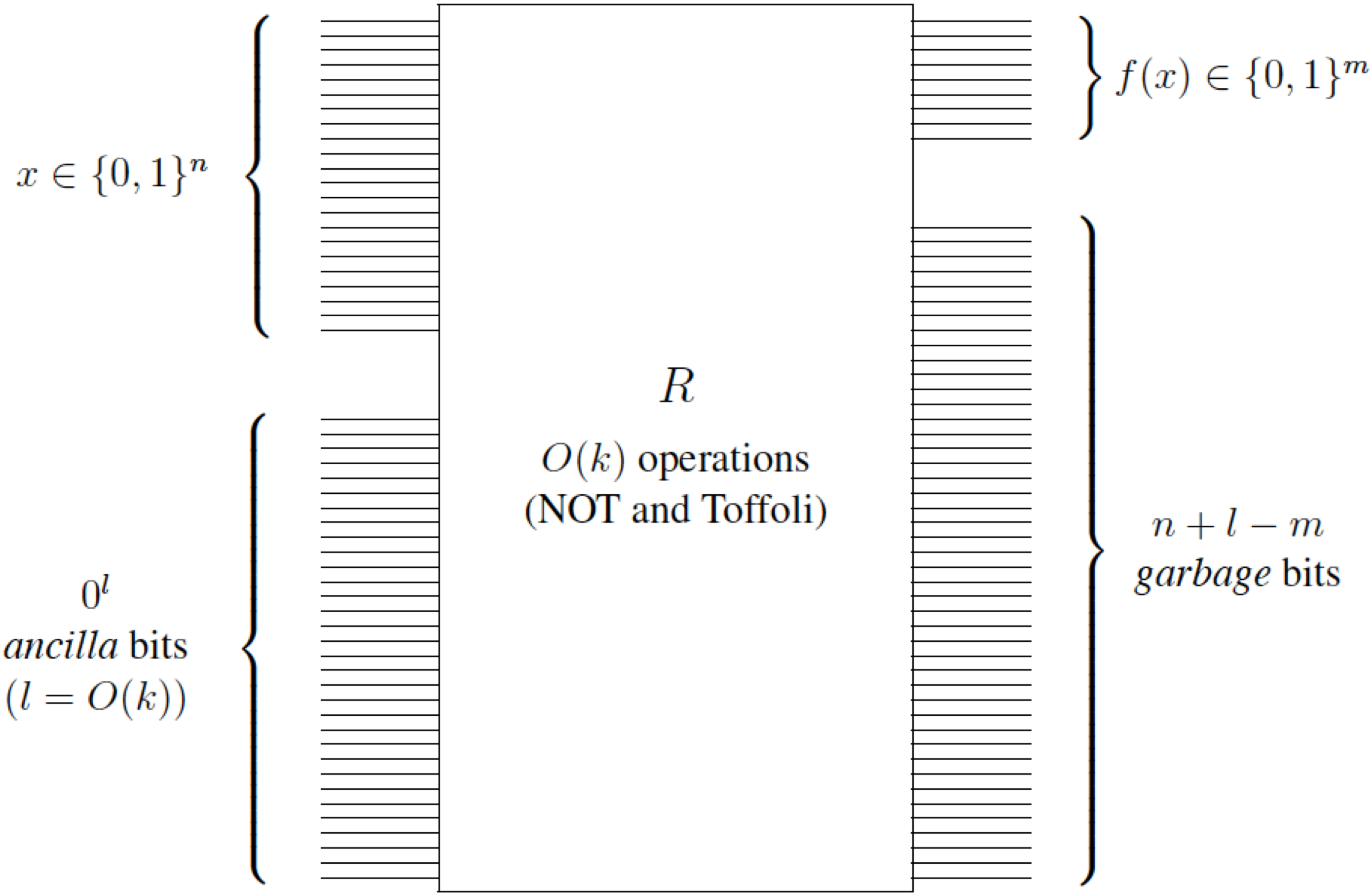
Simulating classical circuits

- All of the operations needed for simulating classical circuits map classical states to classical states
- Such operations can be represented by permutation matrices
- Circuits or operations that always map classical states to classical states are called *reversible* operations or circuits

Simulating classical circuits



Simulating classical circuits



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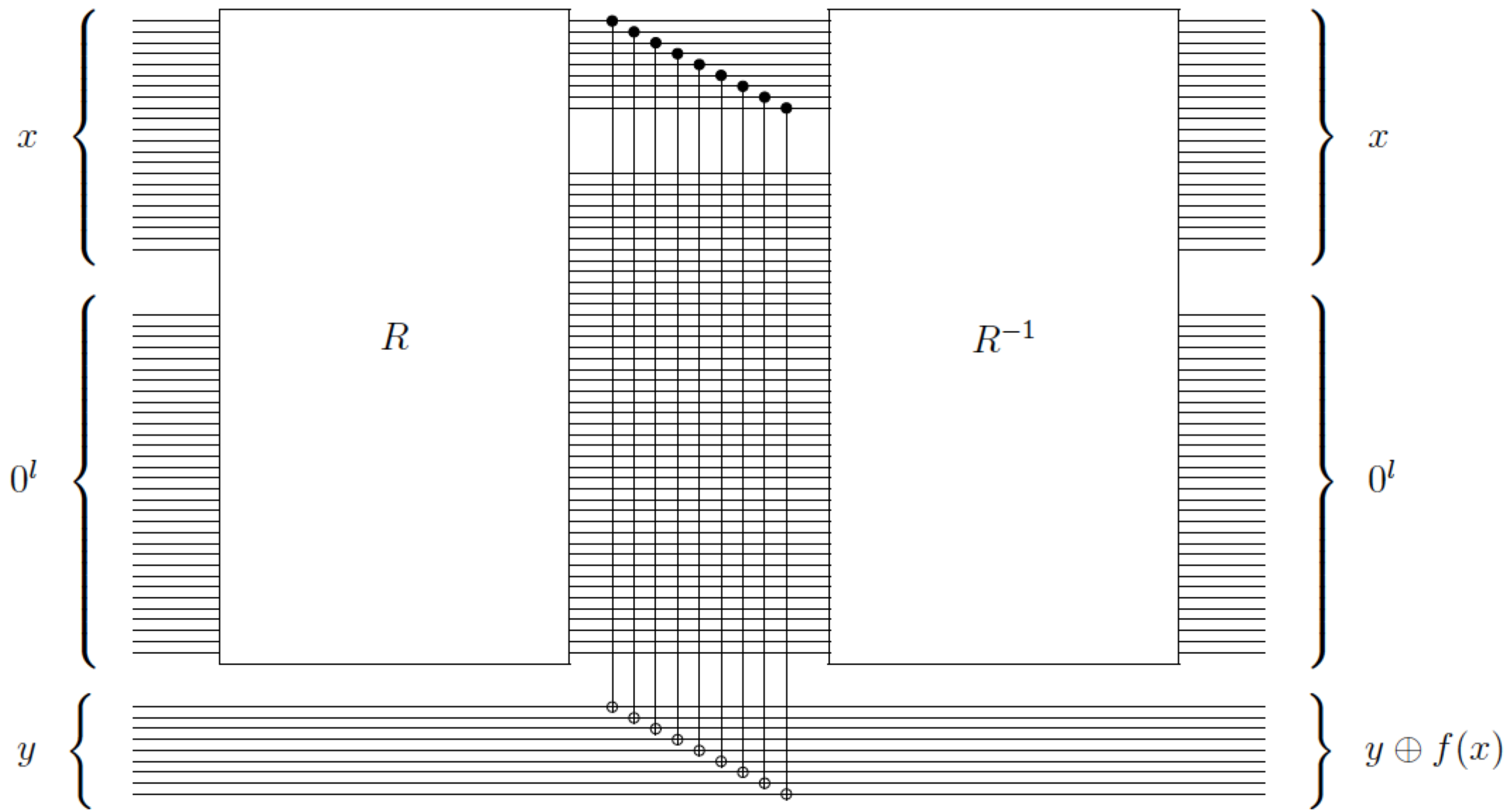

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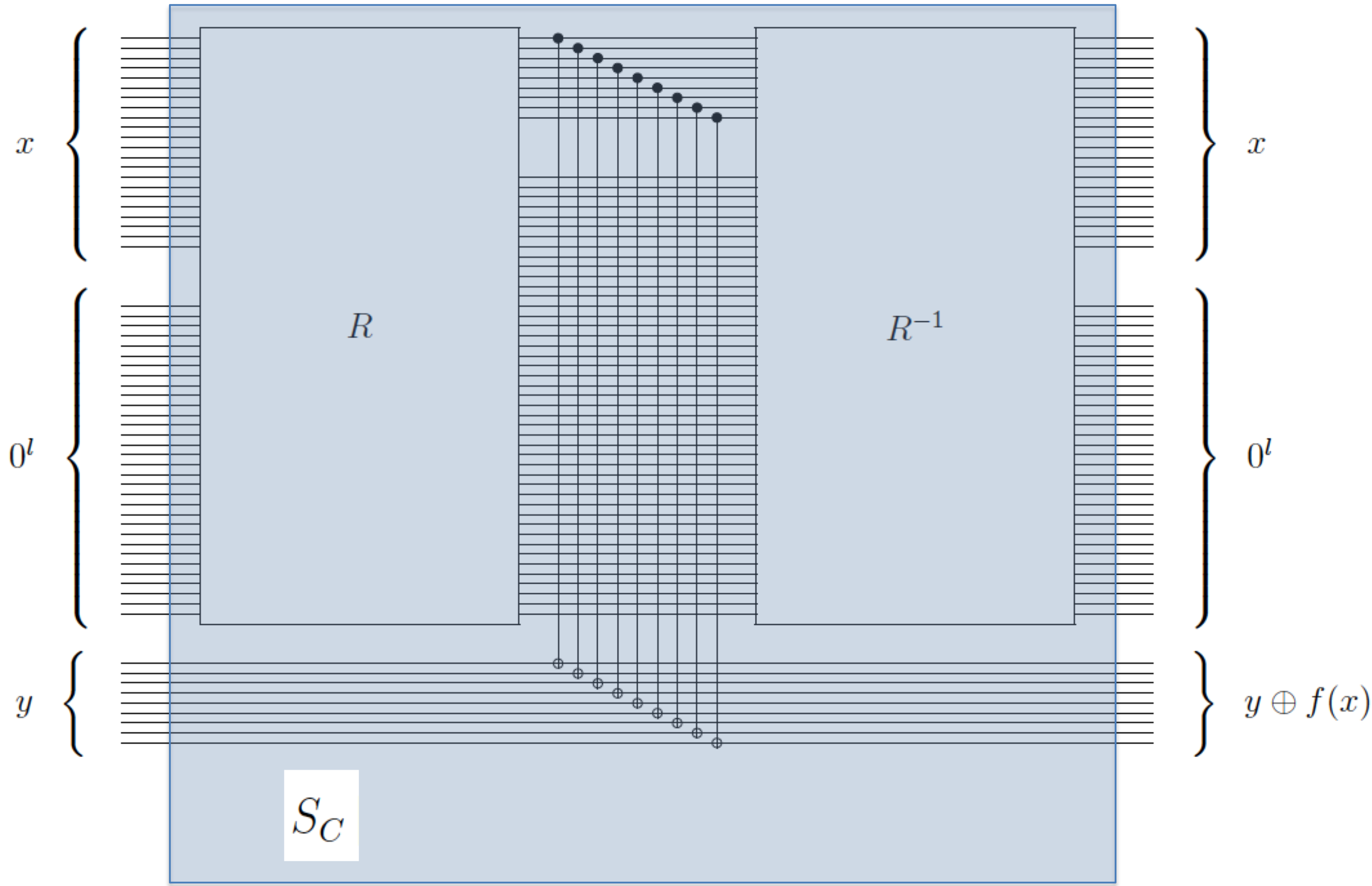

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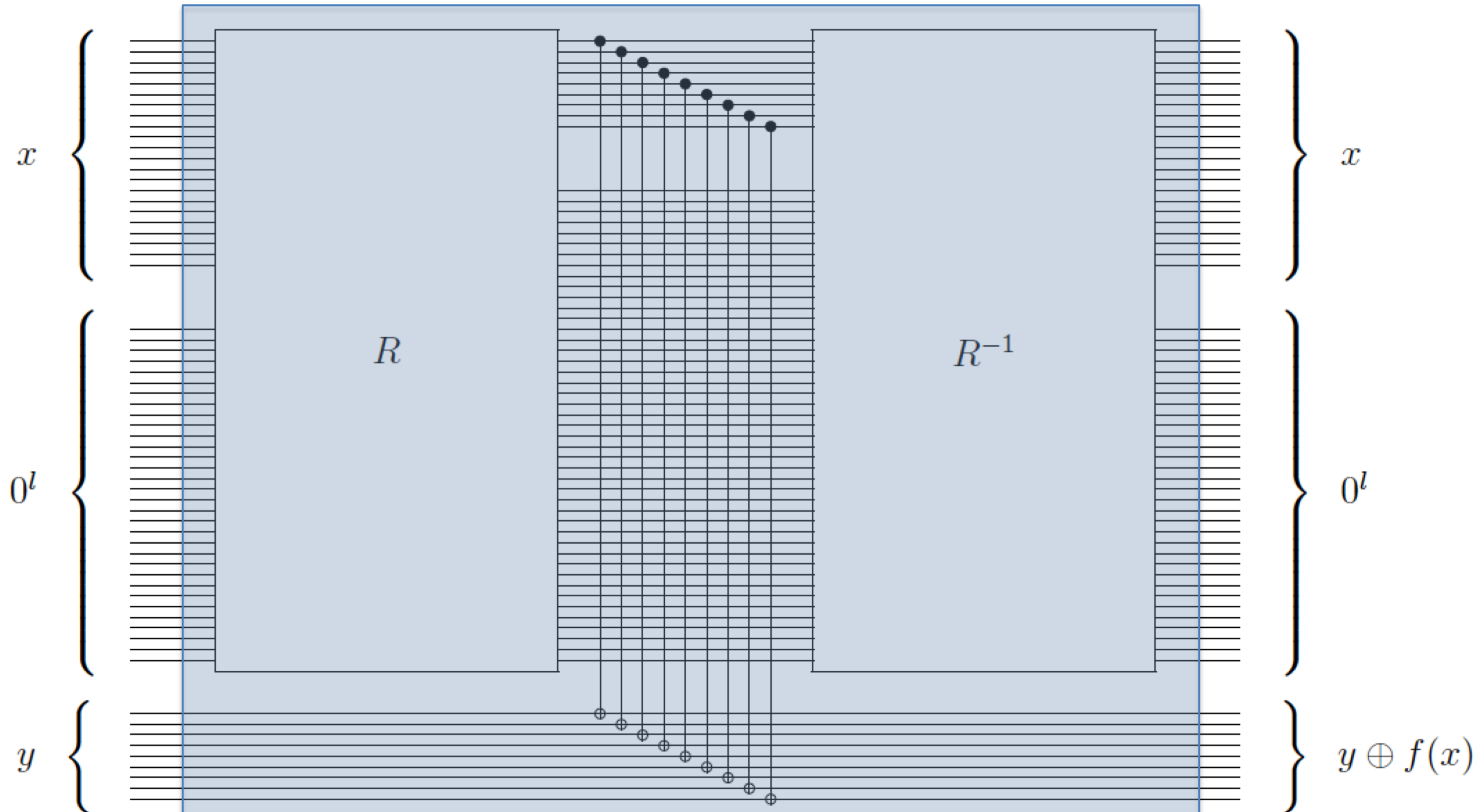

$$S_C \, |x\rangle \, |y\rangle \, |0^l\rangle = |x\rangle \, |y \oplus f(x)\rangle \, |0^l\rangle$$

Summary

- We can describe interesting protocols on quantum bits using quantum circuits, e.g. superdense coding
- Quantum circuits can efficiently simulate classical deterministic circuits if f: $\{0,1\}^n \rightarrow \{0,1\}^m$ can be computed by classical circuit C, then our simulation procedure generates a *reversible* circuit S_C that satisfies

$$S_C |x\rangle |y\rangle |0^l\rangle = |x\rangle |y \oplus f(x)\rangle |0^l\rangle$$

Next class

- Universal quantum circuits
- Quantum complexity classes