# Intro to Quantum Computing

Quantum states, measurements, operations

Dirac notation for representing and manipulating states and operations

See Quantum Computing (not Quantum Information) notes by John Watrous)

# Quantum states

Let X be a physical device with discrete states

For simplicity, let the set of possible states be ∑ = {0,1}, in which case X is a *qubit*

A *quantum state*, or *superposition* is a vector, e.g.,

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{3}{5} \\ \frac{4i}{5} \end{pmatrix}$$

← entry indexed by 0

← entry indexed by 1

# Quantum states

Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be a superposition

Then $\alpha$, $\beta$ are complex numbers called *amplitudes,* and the Euclidian norm of the vector must be 1, i.e., $|\alpha|^2 + |\beta|^2 = 1$

# Quantum measurements

Suppose that the superposition of device X is $\binom{\alpha}{\beta}$

You can measure X, in which case you'll see 0 with probability $|\alpha|^2$ and 1 with probability $|\beta|^2$

After measuring X, its state changes to either

$$\binom{1}{0} \quad \text{or} \quad \binom{0}{1}$$

# Quantum operations

- Operations map superpositions to superpositions, and are represented by matrices that are *unitary*, i.e., preserve the Euclidian norm

- A matrix U is unitary if and only if

  $U^{+}\, U = I$

  where $U^{+}$ is the the *complex conjugate* of U: transpose U, then replace each entry a+*i* b by a-*i* b

- *Exercise*: Check for 2 x 2 matrices why these two definitions are the same

# Quantum operations

- Examples: H below is called the *Hadamard transform*

$$H = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

# Quantum operations

- Example

$$Hv = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- If you measure X after applying the Hadamard transformation, as above, what is the probability of seeing 0? Of seeing 1?
- What is the result of applying H again?

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# Quantum operations

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- If you measure X after applying the Hadamard transformation, as above, what is the probability of seeing 0? Of seeing 1?
- What is the result of applying H again?
- What is the result of applying H to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ?

# Quantum operations

- Example: suppose that qubit X is in one of the following two initial superpositions:

$$v_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{or} \quad v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

- Using operations and measurements, how can you determine which superposition X is in initially? (It's ok to use operations on the superposition, before taking a measurement, in order to determine the answer)

# Multiple qubits

# Multiple qubits

- The superposition of two qubits can be represented as a vector with four entries and Euclidian norm 1

$$\begin{array}{ll} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{2} \\ -\frac{1}{2} \end{pmatrix} & \begin{array}{l} \leftarrow \text{entry indexed by } 00 \\ \leftarrow \text{entry indexed by } 01 \\ \leftarrow \text{entry indexed by } 10 \\ \leftarrow \text{entry indexed by } 11 \end{array} \end{array}$$

- If you measure this quantum state, what is the probability of seeing 00, 01, 10, 11?

# Tensor product of superpositions

# Tensor product of superpositions

Useful for expressing the superpositions of *uncorrelated* states. If X and Y have superpositions

$$v = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

then the superposition of (X,Y) is the *tensor product*

$$v \otimes w = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

# Tensor product of superpositions

Superpositions of *entangled* states X and Y cannot be expressed as a tensor product, e.g.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

# Unitary operations on multiple qubits

- An operation on two bits (X,Y): $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

- To apply the operation, multiply with the superposition:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

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- Suppose that X,Y are uncorrelated and have superpositions $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

- What is the result of the operation? What if X = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

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- What is the result of the operation? What if X = $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$?

- This operation is called the *controlled not* operation

# Tensor product of matrices (operations)

- If operation U transforms X and operation V independently transforms Y then we can express the combined operation on (X,Y) as the tensor product U ⊗ V:

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{NOT} \quad \text{and} \quad V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = H,$$

$$U \otimes V = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}$$

# Tensor product of matrices (operations)

- Then, to calculate the result of the operation U ⊗ V on a superposition v of (X,Y), multiply (U ⊗ V) and v:

$$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

# Tensor product of matrices (operations)

- Given

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \qquad B = \begin{pmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,l} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{k,1} & b_{k,2} & \cdots & b_{k,l} \end{pmatrix}$$

their tensor product is

$$A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{pmatrix}$$

# Tensor product of matrices (operations)

- *Exercise*: Suppose that qubits X and Y are in the superposition :

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Calculate the result of applying the Hadamard transform to X and doing nothing to Y.

# Tensor product properties

- *Associative law*: $(A\otimes B)\otimes C = A\otimes(B\otimes C)$
- *Distributive laws*:

  $A\otimes(B+C) = (A\otimes B) + (A\otimes C)$

  $(A+B)\otimes C = (A\otimes C) + (B\otimes C)$

  $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$
- *Scalar multiplication*: For scalar $\alpha$,

  $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A\otimes B)$

The commutative law $A\otimes B = B\otimes A$ does *not* hold in general

# Dirac notation

- For large state spaces, dirac notation is easier to work with than vectors, matrices and tensor products

# Dirac notation: single bits

- Column vectors are represented by "kets":

$$|0\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- If $|\phi\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix}$ then we can write

$$|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

# Dirac notation: single bits

Applying operations to kets:

- Old notation:

$$Hv = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- Dirac notation:

$$H\left|0\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

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- Dirac notation:

$$H\left|0\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle$$

- Similarly:

$$H\left|1\right\rangle = \frac{1}{\sqrt{2}}\left|0\right\rangle - \frac{1}{\sqrt{2}}\left|1\right\rangle$$

# Dirac notation: multiple bits

- Juxtaposition of kets denotes tensor product:

$$|\psi\rangle\,|\phi\rangle \stackrel{\text{def}}{=} |\psi\rangle \otimes |\phi\rangle$$

- Further shorthand:

$$|01\rangle \stackrel{\text{def}}{=} |0\rangle\,|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

# Dirac notation: multiple bits

- What do you think that $|1010\rangle$ represents?

# Dirac notation: multiple bits

- What do you think that $|1010\rangle$ represents?
- What about $\frac{1}{\sqrt{2}}|000000\rangle + \frac{1}{\sqrt{2}}|111111\rangle$ ?

# Dirac notation: calculations

- *Exercise*: Suppose that pair (X,Y) is in the superposition:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

- Calculate the result of applying the Hadamard transform to X and doing nothing to Y, this time using Dirac notation

# Dirac notation: calculations

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- Recall that

$$H\,|0\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle \qquad H\,|1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{1}{\sqrt{2}}\,|1\rangle$$

# Summary

- Quantum states (e.g, bits) are represented as *superpositions*, just as probabilistic states are represented as probability vectors
- We can do two things to quantum states: measure them, or perform an operation on them
- Quantum operations are represented by unitary matrices

# Summary

- *Tensor products* are handy for combining operations on single bits into operations on multiple bits
- *Dirac notation (*e.g., “ket”*)* is convenient for describing, and operating on, quantum states with multiple qubits

# Dense supercoding premise

- Qubits A and B are entangled in the superposition $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

- Alice holds A and Bob holds B
- Alice also has two (classical) bits a and b

- We'll see how Alice can communicate the two classical bits a and b to Bob using just one qubit, *given that they already share an e-bit* (entangled bit)

# Dense supercoding protocol

1. Alice: If a = 1, apply $\sigma_z$ to qubit A, where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
2. Alice: If b = 1, apply NOT to qubit A
3. Alice: send A to Bob
4. Bob: Apply a controlled-NOT to (A,B) (A is the control)
5. Bob: Apply a Hadamard transform to A
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# Dense supercoding protocol analysis

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| $ab$ | (A,B) after step 1 | (A,B) after step 2 | (A,B) after step 4 | (A,B) after step 5 |
|---|---|---|---|---|
| 00 | | | | |
| 01 | | | | |
| 10 | | | | |
| 11 | | | | |

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| 10 | $\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$ | | | |
| 11 | $\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$ | | | |

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| 01 | $\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$ | $\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$ | | |
| 10 | $\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$ | $\frac{1}{\sqrt{2}}\|00\rangle - \frac{1}{\sqrt{2}}\|11\rangle$ | | |
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| 01 | $\frac{1}{\sqrt{2}}\|00\rangle + \frac{1}{\sqrt{2}}\|11\rangle$ | $\frac{1}{\sqrt{2}}\|10\rangle + \frac{1}{\sqrt{2}}\|01\rangle$ | $\left(\frac{1}{\sqrt{2}}\|1\rangle + \frac{1}{\sqrt{2}}\|0\rangle\right)\|1\rangle$ | |
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# Next Class

- Quantum circuits (lecture 3 of Watrous' notes)