CPSC506 Complexity of Computation Term 2, 2020

Homework # 4

Due on Monday, March 16.

1. Consider the following reduction, which maps instances ϕ of 3SAT to instances ϕ' of 2SAT. For each clause $c = (x \vee y \vee z)$ in ϕ add the following 10 clauses to ϕ' (where w_c is a new variable not in ϕ):

$$(x), (y), (z), (\bar{x} \vee \bar{y}), (\bar{y} \vee \bar{z}), (\bar{x} \vee \bar{z}), (w_c), (x \vee \bar{w_c}), (y \vee \bar{w_c}), (z \vee \bar{w_c}).$$

   Use this reduction, together with the fact that Max 3SAT is hard to approximate and the theorem below on gap-preserving reductions (from the lecture slides) to obtain a hardness of approximation result for Max 2SAT.

   Theorem (from lecture slides, slightly restated): Let Π and Π' be two maximization problems. Given a poly-time gap-preserving reduction from Π to Π' with parameters (c, s) and (c', s'), if it is NP-hard to distinguish instances I of Π with Opt$(I)/|I| \geq c$ from instances with Opt$(I)/|I| < s$ then there is no poly-time approximation algorithm for Π' with ratio c'/s' unless NP=P.

   Here, a gap-preserving reduction from Π to Π' with parameters (c, s) and (c', s') is a function mapping instance I of Π to instance I' of Π' such that

$$\begin{aligned} Opt(I)/|I| \geq c &\implies Opt(I')/|I'| \geq c' \\ Opt(I)/|I| < s &\implies Opt(I')/|I'| < s'. \end{aligned}$$

2. Suppose that all languages in NP have PCP verifiers that for some constants α, c, and q, on inputs of length n, use at most $c\alpha \log n$ random bits, at most $q\alpha \log n$ queries, and have error probability less than $n^{-\alpha}$. Show that for some constant $\epsilon > 0$, if there is a polynomial-time approximation algorithm for Max Clique with approximation ratio n^ϵ then P=NP. You can show this by adapting the PCP-based reduction from languages in NP to Clique presented in class (see slide 4 of lecture 19).

3. In class we saw a simple deterministic greedy 2-approximation algorithm for Max 3SAT. In this problem you will develop a better randomized approximation algorithm for Max 3SAT. Throughout, we assume that our Max 3SAT instances can not only have clauses with one, two or three literals but can also have clauses with zero literals. Such clauses are either **true** or **false**.

   The randomized algorithm works as follows. Given instance ϕ of Max 3SAT, first convert ϕ into a 3CNF instance ϕ' by repeating the following steps until they can no longer be applied:

   - Remove any **false** clause.
   - Replace any pair of clauses z and $\bar{z}$ by a single clause **true**.
   - Replace any three clauses of the form x, y and $\bar{x} \vee \bar{y}$ (where x and y are any literals) by $x \vee y$, **true**.

   Then for each variable x, assign x to be true with probability 2/3 if unit clause x appears in ϕ', assign x to be false with probability 2/3 if unit clause $\bar{x}$ appears in ϕ', and assign x to be true with probability 1/2 otherwise. Output the resulting assignment as the assignment for ϕ.

(a) Show that the expected number of clauses of ϕ' that are satisfied by the above algorithm is at least 2/3.

(b) Show that the maximum number of simultaneously satisfiable clauses in ϕ and ϕ' is the same.

(c) What is the expected approximation ratio of this algorithm on ϕ?

(Note: This randomized algorithm can be made deterministic using a technique called the method of conditional expectations.)

4. Deterministic and nondeterministic Oracle Turing machines are defined in section 3.6 of the Arora-Barak text. You should read these definitions in order to do this problem.

   For complexity class $\mathcal{C}$, let $\mathrm{P}^{\mathcal{C}} = \cup_{O \in \mathcal{C}} \mathrm{P}^{O}$ and define $\mathrm{NP}^{\mathcal{C}}$ similarly.

   (a) Show that $\mathrm{NP}^{\mathrm{NP}} = \Sigma_2^p$ (i.e., the second level of the polynomial hierarchy).

   (b) Show that P^{NP} is contained in $\Sigma_2^p \cap \Pi_2^p$.

   (c) Provide a definition of the class $\mathrm{RP}^{\mathrm{NP}}$. Find complexity classes $\mathcal{C}$ and $\mathcal{C}'$ such that $\mathcal{C} \subseteq \mathrm{RP}^{\mathrm{NP}} \subseteq \mathcal{C}'$.