CPSC506 **Complexity of Computation** **Term 2, 2020**

Homework # 1
Due on Monday, Jan 27, 2020

General guidelines for homeworks

I encourage you to discuss the problems with others in the class, but do all write-ups on your own. Homework grades will be based not only on getting the correct answer, but also on good writing style and clear presentation of your solution. Handwritten homeworks are acceptable if written clearly, but if you have poor handwriting, please type up your solutions, preferably using latex or tex.

It is not always easy to gauge what is the right level of detail to include. Please try to keep your proofs concise, while still including enough information to show that you have thought through any subtle issues. If your work is lengthy, read it over and see if you can find a more elegant way to present it.

At the top of your homework, please acknowledge the people with whom you discussed the problems and any resources that you consulted.

You can email me (condoncs.ubc.ca) a pdf of your solutions by end of day on the due date, or provide me with a hard copy in class.

Problems

1. A *succinct representation of a Boolean formula* in conjunctive normal form (CNF) with n variables and m clauses is a Boolean circuit C which:

   - On input $(0, i, k)$ returns returns the index of the clause where the literal $\bar{x}_i$ (i.e., the negation of variable x_i) appears for the kth time (and returns 0 if literal $\bar{x}_i$ appears in fewer than k clauses).
   - On input $(1, i, k)$ returns the index of the clause where the literal x_i appears for the kth time (and returns 0 if literal x_i appears in fewer than k clauses).

   The Succinct SAT probem is:

   > **Instance**: Positive integers n, m, and a succinct representation of a CNF formula ϕ.
   > **Problem**: Is ϕ satisfiable?

   Adapt the Cook-Levin theorem that SAT is NP-complete to show that Succinct SAT is NEXP-complete. (I'll put a proof of that theorem on the course webpage, along with this homework.)

2. Prove the Space Hierarchy theorem, which is Theorem 3.2 of Arora and Barak: If f and g are space-constructible functions satisfying $f(n) = o(g(n))$, then SPACE($f(n)$) $\subsetneq$ SPACE($g(n)$).

3. Recall the Wang Tiling Problem, introduced in the first lecture:

   Instance: A finite set T of Wang tiles: square tiles, each with a fixed orientation, edges labeled with colours.

   Problem: Is there an infinite tiling of the plane, using tiles in T?

   Here, an infinite tiling of the plane is a mapping from $\mathbb{Z}^2$ to T such that colours on adjacent tiles match, where tiles must be in their fixed orientation (no rotations or reflections). Wang conjectured that there is an infinite tilings of the plane using tiles in T, then T allows a *periodic* tiling of the plane, i.e., a tiling with infinitely repeatable square of tiles, or "period". In class we developed the following algorithm to show that if Wang's conjecture is true, then the Wang Tiling problem is decidable.

```
procedure WANG-TILING(T)
    i = 1
    repeat
        if there is no valid i × i tiling then
            return "no"
        else
            for every valid i × i tiling do
                if the tiling is a period then return "yes"
                end if
            end for
        end if
    until Forever
end procedure
```

   Prove that if Wang's conjecture were true, then this algorithm would be correct. The following lemma might be useful:

   Lemma 1 *(König's Infinity Lemma) Let G be a connected graph with an infinite number of nodes, each of which is connected to a finite number of other nodes. Then G contains a path with an infinite number of nodes that has no repeated vertices.*

4. The abstract Tile Assembly Model (aTAM) is an algorithmic variant of the Wang Tiling model that is useful both for designing DNA tile assembly systems and for understanding their capabilities. Like Wang tiles, aTAM tiles are square, with coloured edges and a fixed orientation (they cannot be rotated or flipped). Unlike Wang tiles, each color also has a glue strength which is a nonnegative integer.

   Let T be a finite set of tiles and let $s \in T$ be a designated seed tile. An *assembly* is a partial function a that maps a finite, connected subset S of $\mathbb{Z}^2$ containing (0,0) to T, with $(0,0)$ mapped to s. Here, S is *connected* if for every two distinct pairs (i, j) and (i', j') in S there is a sequence

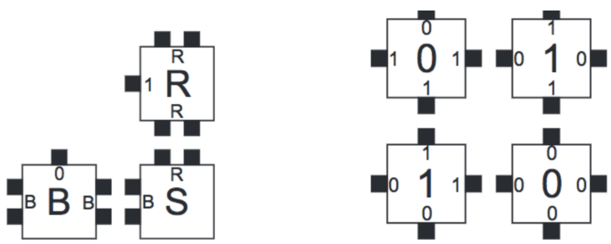
$$(i, j) = (i_0, j_0), (i_1, j_1), \ldots, (i_k, j_k) = (i', j')$$

   such that for all $l, 0 < l < k$, $(i_l, j_l) \in S$ and moreover, $|i_l - i_{l+1}| + |j_l + j_{l+1}| = 1$.

   Positions in S are *tiled positions* and the remaining positions are *untiled*. If two adjacent tile edges have the same colour, their glue strength is that of the colour, and if they have different colours then their glue strength is zero.

Formally, an aTAM system is a triple (T, s, τ) where T is a finite set of aTAM tile types containing s, and τ is a positive integer. Assemblies of the system form when tiles "self assemble" in an asynchronous, nondeterministic fashion, building from the seed, in a manner that is controlled by glue strengths. Initially, the assembly consists only of s at position $(0, 0)$. A tile of type $t \in T$ can be added to an untiled position of the assembly if the sum of glue strengths between edges of t and adjacent edges of tiles already in the assembly is at least τ. Unlike Wang tilings, colours on two adjacent edges of tiles in an assembly need not match. An assembly obtained in this way is called *τ-stable*.

(a) Explain why self-assembly of the set of tiles in the figure below, where the tile labeled "S" is the seed, simulates an infinite binary counter when $\tau = 2$, when one considers the arrangement of tile labels. (The figure is from "An Introduction to Tile-Based Self-Assembly Systems and a Survey of Recent Results" by Matthew Patitz.)



(a) The tile types which form the border of the counter

(b) The "rule" tile types which compute and represent the values of the counter

(b) Design an aTAM $(T, s, 2)$ that self-assembles into an $n \times n$ square, where the size of T is $\Theta(n)$.

5. Do either part (a) or part (b):

   (a) Show that the following problem is undecidable: Given an aTAM system $(T, s, 2)$, and a tile type t in T, is there a (2-stable) assembly that includes t?

   (b) One thread of research on aTAMs explores the computational complexity of shapes, measured by the size of the set of tile types needed for the shape to form via aTAM self-assembly. Design an aTAM $(T, s, 2)$ that self-assembles into an $n \times n$ square, where the size of T is $O(\log n)$.